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Homework 7 solutions

1. Solve the problem

$$\begin{aligned} u_{tt} - 4u_{xx} &= 0 && \text{if } 0 < x < 1, t > 0, \\ u_x(0,t) = u_x(1,t) &= 0 && \text{if } t \geq 0, \\ u(x,0) &= 2\cos(3\pi x) + 4 && \text{if } 0 \leq x \leq 1, \\ u_t(x,0) &= \sin^2(\pi x) - \cos^2(\pi x) && \text{if } 0 \leq x \leq 1. \end{aligned}$$

You may use the general solution derived in the professor's lecture notes.

Solution. The general solution derived in the professor's lecture notes (at the beginning of Lecture 7, on April 22) with $c = 2$ and $L = 1$ is

$$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} \cos(n\pi x)(A_n \cos(2n\pi t) + B_n \sin(2n\pi t)).$$

Its first partial derivative with respect to t is

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\left(\frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} \cos(n\pi x)(A_n \cos(2n\pi t) + B_n \sin(2n\pi t)\right) \\ &= \frac{1}{2}\frac{\partial}{\partial t}(A_0 + B_0 t) + \sum_{n=1}^{\infty} \cos(n\pi x)\frac{\partial}{\partial t}(A_n \cos(2n\pi t) + B_n \sin(2n\pi t)) \\ &= \frac{B_0}{2} + \sum_{n=1}^{\infty} 2n\pi \cos(n\pi x)(-A_n \sin(2n\pi t) + B_n \cos(2n\pi t)). \end{aligned}$$

Now, we recall the given initial conditions

$$u(x,0) = 2\cos(3\pi x) + 4,$$

and

$$\begin{aligned} u_t(x,0) &= \sin^2(\pi x) - \cos^2(\pi x) \\ &= -\cos(2\pi x) \end{aligned}$$

where in the last step above we have employed the triple-angle trigonometric identity $\cos(2\theta) = \sin^2(\theta) - \cos^2(\theta)$. Also, at $t = 0$, our solution becomes

$$\begin{aligned} u(x,0) &= \frac{1}{2}(A_0 + B_0(0)) + \sum_{n=1}^{\infty} \cos(n\pi x)(A_n \cos(2n\pi(0)) + B_n \sin(2n\pi(0))) \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \\ &= \frac{A_0}{2} + A_1 \cos(1\pi x) + A_2 \cos(2\pi x) + A_3 \cos(3\pi x) + \sum_{n=4}^{\infty} A_n \cos(n\pi x), \end{aligned}$$

and the partial derivative of our solution becomes

$$\begin{aligned} u_t(x,0) &= \frac{B_0}{2} + \sum_{n=1}^{\infty} 2n\pi \cos(n\pi x)(-A_n \sin(2n\pi(0)) + B_n \cos(2n\pi(0))) \\ &= \frac{B_0}{2} + \sum_{n=1}^{\infty} 2n\pi B_n \cos(n\pi x) \\ &= \frac{B_0}{2} + 1\pi B_1 \cos(1\pi x) + 2\pi B_2 \cos(2\pi x) + \sum_{n=3}^{\infty} 2n\pi B_n \cos(n\pi x). \end{aligned}$$

Both our expressions of $u(x,0)$ yield

$$\frac{A_0}{2} + A_1 \cos(1\pi x) + A_2 \cos(2\pi x) + A_3 \cos(3\pi x) + \sum_{n=4}^{\infty} A_n \cos(n\pi x) = 2\cos(3\pi x) + 4.$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to obtain the Fourier coefficients

$$\begin{aligned} A_0 &= 8, \\ A_3 &= 2, \\ A_n &= 0 \end{aligned}$$

for $n = 1, 2$ and for $n = 4, 5, 6, \ldots$. Similarly, both our expressions of $u_t(x, 0)$ yield

$$\frac{B_0}{2} + 1\pi B_1 \cos(1\pi x) + 2\pi B_2 \cos(2\pi x) + \sum_{n=3}^{\infty} 2n\pi B_n \cos(n\pi x) = -\cos(2\pi x).$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to obtain the Fourier coefficients

$$\begin{aligned} B_2 &= -\frac{1}{4\pi}, \\ B_n &= 0 \end{aligned}$$

for $n = 0, 1$ and for $n = 3, 4, 5, \ldots$. Therefore, our formal solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} \cos(n\pi x)(A_n \cos(2n\pi t) + B_n \sin(2n\pi t)) \\ &= \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\cos(2n\pi t) + \sum_{n=1}^{\infty} B_n \cos(n\pi x)\sin(2n\pi t)) \\ &= \frac{1}{2}(A_0 + B_0 t) + \left(A_3 \cos(3\pi x)\cos(2(3)\pi t) + \sum_{\substack{n=1,2 \\ n=4,5,6,\ldots}} A_n \cos(n\pi x)\cos(2n\pi t) \right) \\ &\quad + \left(B_2 \cos(2\pi x)\sin(2(2)\pi t) + \sum_{\substack{n=1 \\ n=3,4,5,\ldots}} B_n \cos(n\pi x)\sin(2n\pi t) \right) \\ &= \frac{1}{2}(8 + 0t) + \left(2\cos(3\pi x)\cos(6\pi t) + \sum_{\substack{n=1,2 \\ n=4,5,6,\ldots}} 0\cos(n\pi x)\cos(n\pi t) \right) \\ &\quad + \left(-\frac{1}{4\pi}\cos(2\pi x)\sin(4\pi t) + \sum_{\substack{n=1 \\ n=3,4,5,\ldots}} 0\cos(n\pi x)\sin(n\pi t)) \right) \\ &= \boxed{4 + 2\cos(3\pi x)\cos(6\pi t) - \frac{1}{4\pi}\cos(2\pi x)\sin(4\pi t)}, \end{aligned}$$

as desired. □