Homework 8 solutions

Note: I am re-deriving here the eigenvalues and eigenfunctions in order to write a complete solution. But you do not need to do this in your own solutions because the professor has already done this in your lecture notes.

1. Solve the problem

$$\begin{aligned} u_t - 4u_{xx} &= \sin(t) && \text{if } 0 < x < 1, t > 0,\\ u_x(0,t) = u_x(1,t) &= 0 && \text{if } t \geq 0,\\ u(x,0) &= 1 + \cos^2(\pi x) && \text{if } 0 \leq x \leq 1. \end{aligned}$$

Solution. First, we need to find all the eigenvalues and eigenfunctions of the homogeneous problem

$$\begin{aligned} u_t - 4u_{xx} &= 0 && \text{if } 0 < x < 1, t > 0,\\ u_x(0,t) = u_x(1,t) &= 0 && \text{if } t \geq 0,\\ u(x,0) &= 1 + \cos^2(\pi x) && \text{if } 0 \leq x \leq 1. \end{aligned}$$

To do this, we can proceed as we did in the method of separation of variables by writing

$$u(x,t) = X(x)T(t).$$

Our partial derivatives are

$$\begin{aligned} u_t(x,t) &= X(x)T_t(t),\\ u_{xx}(x,t) &= X_{xx}(t)T(t). \end{aligned}$$

So the partial differential equation

$$u_t - 4u_{xx} = 0$$

becomes

$$X(x)T_t(t) - 4X_{xx}(x)T(t) - hX(x)T(t) = 0,$$

which we can algebraically rearrange to write

$$\frac{X_{xx}(x)}{X(x)} = \frac{T_t(t)}{4T(t)} = -\lambda,$$

where λ is a constant in both x and t. This produces the system of two ordinary differential equations

$$\begin{aligned} \frac{d^2X}{dx^2} + \lambda X &= 0\\ \frac{dT}{dt} + 4\lambda T &= 0. \end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$\begin{aligned} X(x) &= \begin{cases} C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0,\\ C_1 x + C_2 & \text{if } \lambda = 0,\\ C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x) & \text{if } \lambda > 0, \end{cases}\\ T(t) &= De^{-4\lambda t} \end{aligned}$$

where C_1, C_2, D are constants. Now, the boundary conditions

$$u_x(0,t) = u_x(1,t) = 0$$

are equivalent to

$$\begin{aligned} \frac{d}{dx}X(0)T(t) &= 0,\\ \frac{d}{dx}X(1)T(t) &= 0, \end{aligned}$$

which imply either $T(t) = 0$ or $\frac{d}{dx}X(0) = \frac{d}{dx}X(1) = 0$. If $T(t) = 0$, then we would have

$$\begin{aligned} u(x,t) &= X(x)T(t)\\ &= X(x)0\\ &= 0, \end{aligned}$$

which would be a trivial solution. So we should assume

$$\frac{d}{dx}X(0) = \frac{d}{dx}X(1) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then we have

$$\begin{aligned}X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x},\\ X(0) &= 0,\end{aligned}$$

which implies $C_1 + C_2 = 0$, or $C_2 = -C_1$. So we have

$$\begin{aligned}X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}\\ &= C_1 e^{\sqrt{-\lambda}x} - C_1 e^{-\sqrt{-\lambda}x}\\ &= C_1(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}).\end{aligned}$$

We notice $e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L} \neq 0$ unless $\lambda = 0$. This means

$$\begin{aligned}X(x) &= C_1(e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}),\\ X(L) &= 0\end{aligned}$$

implies $C_1 = 0$, and so we have

$$\begin{aligned}X(x) &= C_1(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x})\\ &= 0(e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x})\\ &= 0,\end{aligned}$$

which would mean u is a trivial solution. Therefore, the problem has no negative eigenvalues.

- Case 2: Suppose $\lambda = 0$. Then we have

$$\begin{aligned}X(x) &= C_1 x + C_2,\\ \frac{d}{dx}X(0) &= 0,\end{aligned}$$

which implies

$$\begin{aligned}\frac{d}{dx}X(x) &= C_1,\\ \frac{d}{dx}X(0) &= 0,\end{aligned}$$

which implies $C_1 = 0$, and so we have

$$\begin{aligned}X(x) &= C_1 x + C_2\\ &= 0x + C_2\\ &= C_2,\end{aligned}$$

which already satisfies $\frac{d}{dx}X(1) = 0$. Therefore, if we let $\frac{A_0}{2} = C_2 D$, then we have

$$\begin{aligned}u_0(x,t) &= X_0(x)T_0(t)\\ &= C_2 D e^{-\lambda\cdot 0}\\ &= C_2 D\\ &= \frac{A_0}{2},\end{aligned}$$

which is a nontrivial solution.

- Case 3: Suppose $\lambda > 0$. Then we have

$$\begin{aligned}X(x) &= C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x),\\ \frac{d}{dx}X(0) &= 0,\end{aligned}$$

which implies

$$\begin{aligned}\frac{d}{dx}X(x) &= \sqrt{\lambda}(-C_1\sin(\sqrt{\lambda}x) + C_2\cos(\sqrt{\lambda}x)),\\ \frac{d}{dx}X(0) &= 0,\end{aligned}$$

which implies $C_2 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \\ &= C_1 \cos(\sqrt{\lambda}x) + 0 \sin(\sqrt{\lambda}x) \\ &= C_1 \cos(\sqrt{\lambda}x). \end{aligned}$$

Next, we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda}x), \\ \frac{d}{dx}X(1) &= 0 \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dx}X(x) &= -\sqrt{\lambda}C_1 \sin(\sqrt{\lambda}x), \\ \frac{d}{dx}X(1) &= 0 \end{aligned}$$

implies either $C_1 = 0$ or $\sin(\sqrt{\lambda}) = 0$. But $C_2 = 0$ (with $C_1 = 0$) implies $X(x) = 0$ and that $u(x,t)$ would be a trivial solution. So we should assume $\sqrt{\lambda} = n\pi$, or equivalently the eigenvalues

$$\lambda_n = \lambda = (n\pi)^2,$$

with the corresponding eigenfunctions

$$\begin{aligned} X_n(x) &= C_{1,n} \cos(\sqrt{\lambda_n}x) \\ &= C_{1,n} \cos(\sqrt{(n\pi)^2}x) \\ &= C_{1,n} \cos(n\pi x), \end{aligned}$$

as desired.

From the three cases above, we see that the problem has the zero eigenvalue $\lambda = 0$ and its corresponding eigenfunction $X_0(x) = C_2$, as well as positive eigenvalues $\lambda_n = (n\pi)^2$ and their corresponding eigenfunctions $X_n(x) = C_{1,n}\cos(n\pi x)$ (or just $X_n(x) = \cos(n\pi x)$; these two eigenfunctions are the same up to a scaling factor). We will now use the method of eigenfunction expansion. Based on our eignefunction $X_n(x) = \cos(n\pi x)$, we can represent, for any fixed t, our solution as

$$u(x,t) = \frac{1}{2}T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(n\pi x),$$

where $T_n(t)$ for $n = 1, 2, 3, \ldots$ are the time-dependent Fourier coefficients. Our derivatives are

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\left(\frac{1}{2}T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(n\pi x)\right) \\ &= \frac{1}{2}T_0'(t) + \sum_{n=1}^{\infty} T_n'(t)\cos(n\pi x) \end{aligned}$$

and

$$\begin{aligned} u_{xx}(x,t) &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}T_0'(t) + \sum_{n=1}^{\infty} T_n'(t)\cos(n\pi x)\right) \\ &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}T_0'(t)\right) + \sum_{n=1}^{\infty} T_n(t)\frac{\partial^2}{\partial x^2}\cos(n\pi x) \\ &= 0 + \sum_{n=1}^{\infty} -(n\pi)^2 T_n(t)\cos(n\pi x) \\ &= \sum_{n=1}^{\infty} -(n\pi)^2 T_n(t)\cos(n\pi x). \end{aligned}$$

So the nonhomogeneous partial differential equation

$$u_t - 4u_{xx} = \sin(t)$$

becomes

$$\left(\frac{1}{2}T_0'(t) + \sum_{n=1}^{\infty} T_n'(t)\cos(n\pi x)\right) - 4\left(\sum_{n=1}^{\infty} -(n\pi)^2 T_n(t)\cos(n\pi x)\right) = \sin(t),$$

or equivalently

$$\frac{1}{2}T_0'(t) + \sum_{n=1}^{\infty}(T_n'(t) + 4(n\pi)^2 T_n(t))\cos(n\pi x) = \sin(t) + 0\cos(n\pi x).$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to obtain the ordinary differential equations

$$\begin{aligned}\frac{1}{2}T_0'(t) - \sin(t) &= 0,\\ T_n'(t) + 4(n\pi)^2 T_n(t) &= 0,\end{aligned}$$

whose general solutions are, respectively,

$$\begin{aligned}T_0(t) &= A_0 - 2\cos(t)\\ T_n(t) &= A_n e^{-4(n\pi)^2 t},\end{aligned}$$

where A_0 and A_n for $n = 1, 2, 3, \ldots$ are the Fourier coefficients. Therefore, our solution is

$$\begin{aligned}u(x,t) &= \frac{1}{2}T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(n\pi x)\\ &= \frac{1}{2}(A_0 - 2\cos(t)) + \sum_{n=1}^{\infty} A_n e^{-4(n\pi)^2 t}\cos(n\pi x).\end{aligned}$$

Now, we can use the given initial conditions to write

$$\begin{aligned}u(x,0) &= 1 + \cos^2(\pi x)\\ &= 1 + \frac{1}{2}(1 + \cos(2\pi x))\\ &= \frac{3}{2} + \frac{1}{2}\cos(2\pi x),\end{aligned}$$

where in the last step above we have employed the double-angle trigonometric identity $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$. Also, at $t = 0$, our solution becomes

$$\begin{aligned}u(x,0) &= \frac{A_0 - 2}{2} + \sum_{n=1}^{\infty} A_n\cos(n\pi x)\\ &= \frac{A_0 - 2}{2} + A_1\cos(\pi x) + A_2\cos(2\pi x) + \sum_{n=3}^{\infty} A_n\cos(n\pi x).\end{aligned}$$

Both our expressions of $u(x,0)$ yield

$$\frac{A_0 - 2}{2} + A_1\cos(\pi x) + A_2\cos(2\pi x) + \sum_{n=3}^{\infty} A_n\cos(n\pi x) = \frac{3}{2} + \frac{1}{2}\cos(2\pi x).$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to obtain the Fourier coefficients

$$\begin{aligned}A_0 &= 5,\\ A_2 &= \frac{1}{2},\\ A_n &= 0\end{aligned}$$

for $n = 1$ and for $n = 3, 4, 5, \ldots$. Therefore, our formal solution is

$$\begin{aligned}u(x,t) &= \frac{1}{2}(A_0 - 2\cos(t)) + \sum_{n=1}^{\infty} A_n e^{-4(n\pi)^2 t}\cos(n\pi x)\\ &= \frac{1}{2}(A_0 - 2\cos(t)) + A_1 e^{-4(1\pi)^2 t}\cos(\pi x) + A_2 e^{-4(2\pi)^2 t}\cos(2\pi x) + \sum_{n=3}^{\infty} A_n e^{-4(n\pi)^2 t}\cos(n\pi x)\\ &= \frac{1}{2}(5 - 2\cos(t)) + 0e^{-4(1\pi)^2 t}\cos(\pi x) + \frac{1}{2}e^{-4(2\pi)^2 t}\cos(2\pi x) + \sum_{n=3}^{\infty} 0e^{-4(n\pi)^2 t}\cos(n\pi x)\\ &= \boxed{\frac{1}{2}(e^{-16\pi^2 t}\cos(2\pi x) + 5) - \cos(t)},\end{aligned}$$

as desired. ◻