Ryan Ta
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Homework 9 solutions

1. The general form for a homogeneous third-degree polynomial is

$$P_3(x,y) = ax^3 + bx^2y + cxy^2 + dy^3,$$

where a, b, c, d are constants.

(i) Find an explicit polynomial that is also a harmonic function; that is, give an example of some $P_3(x,y)$ that satisfies

$$\Delta P_3(x,y) = 0.$$

Solution. Given $P_3(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$, we have

$$\begin{aligned}
\Delta P_3(x,y) &= \frac{\partial^2}{\partial x^2}P_3(x,y) + \frac{\partial^2}{\partial x^2}P_3(x,y) \\
&= \frac{\partial^2}{\partial x^2}(ax^3 + bx^2y + cxy^2 + dy^3) + \frac{\partial^2}{\partial y^2}(ax^3 + bx^2y + cxy^2 + dy^3) \\
&= (6ax + 2by + 0 + 0) + (0 + 0 + 2cx + 6dy) \\
&= 2(3a + c)x + 2(b + 3d)y \\
&= 0,
\end{aligned}$$

provided that we choose the constants a, b, c, d that satisfy

$$\begin{aligned}
3a + c &= 0, \\
b + 3d &= 0.
\end{aligned}$$

For example, if we choose $a = 1$, $b = 3$, $c = -3$, $d = -1$, then the polynomial

$$\begin{aligned}
P_3(x,y) &= ax^3 + bx^2y + cxy^2 + dy^3 \\
&= 1x^3 + 3x^2y + (-3)xy^2 + (-1)y^3 \\
&= x^3 + 3x^2y - 3xy^2 - y^3,
\end{aligned}$$

then this choice of $P_3(x,y)$ satisfies $\Delta P_3(x,y) = 0$ and is therefore harmonic. □

(ii) What is the dimension of the space of all homogeneous third-degree polynomials that are also harmonic functions?

Solution. Let A be the space of all homogeneous third-degree polynomials that are also harmonic functions. We will say without proving here that A is a vector space with respect to the operations of adding and multiplying polynomials. Due to the constraints $3a + c = 0$ and $b + 3d = 0$ as seen in our solution to part (i), or equivalently

$$\begin{aligned}
c &= -3a, \\
b &= -3d,
\end{aligned}$$

we have

$$\begin{aligned}
P_3(x,y) &= ax^3 + bx^2y + cxy^2 + dy^3 \\
&= ax^3 + (-3d)x^2y + (-3a)xy^2 + dy^3 \\
&= a(x^3 - 3xy^2) + d(y^3 - 3x^2y).
\end{aligned}$$

In other words, any $P_3(x,y)$ that is also harmonic must be a linear combination of the homogeneous polynomials $x^3 - 3xy^2$ and $y^3 - 3x^2y$. This is equivalent to saying that the set $\{x^3 - 3xy^2, y^3 - 3x^2y\}$ spans A. Also, if we set $P_3(x,y) = 0$, then the only scalars a, d that satisfy

$$a(x^3 - 3xy^2) + d(y^3 - 3x^2y) = 0$$

are $a = 0$ and $d = 0$. This is equivalent to saying that the set $\{x^3 - 3xy^2, y^3 - 3x^2y\}$ is linearly independent. Therefore, the set $\{x^3 - 3xy^2, y^3 - 3x^2y\}$ is a basis of A. Finally, as the dimension of A is determined by the size of the basis of A, we conclude that the dimension of A is 2. □

Remark. *Part (ii) is worth only 1 point. You do not need to justify part (ii) like I did here. If you just wrote down the answer of* 2 *without showing any work, you will get the full credit of 1 point.*