Lecture1

§1.1 – §1.3

Monday, March 30, 2020 7:29 PM

Objective { def, classification of PDE; Superposition principle.

(I) Notations

- multi-variable function
  $$f(x_1, x_2, \cdots, x_n)$$
- partial derivatives:
  $$\frac{\partial f}{\partial x_1} = f_{x_1}$$
- gradient
  $$\nabla f = f_{x_1}\vec{e_1} + f_{x_2}\vec{e_2} + \cdots + f_{x_n}\vec{e_n}$$
  $$\vec{e_i} = (0, \cdots, 0, 1, 0, \cdots 0)$$
  (1 at the i-th position) basis vector in $\mathbb{R}^n$

① $r = \sqrt{x^2+y^2}$ find ∇r

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$

$$\nabla r = \begin{pmatrix} \frac{\partial r}{\partial x} \\ \frac{\partial r}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \end{pmatrix}$$

- Differential operator (in 3-D)

Nabla, $\nabla = \vec{e_x}\frac{\partial}{\partial x} + \vec{e_y}\frac{\partial}{\partial y} + \vec{e_z}\frac{\partial}{\partial z}$

Laplacian $\Delta = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\vec{F} = (F_1, F_2, F_3)$$

$$\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Gauss's Theorem (Thm) (Green's)

$$\int_D \nabla\cdot\vec{F}\, dV = \int_{\partial D} \vec{F}\cdot\vec{n}\, dS$$

$D \subset \mathbb{R}^3$

(II) PDE

Def 1: A Partial Differential Equation (PDE) is an equation between unknown function and its partial derivatives

General form

(*) $F(x_1, x_2, \cdots, u, u_{x_1}, u_{x_2}, \cdots) = 0$

where $u(x_1, x_2, \cdots)$ is unknown.

Analysis of PDE { solving analytically (19th); solving numerically; theoretical analysis

- well-posedness
  1. Existence. at least one solution
  2. Uniqueness. No more than one solution
  3. stability: A small change in PDE system leads to small change in solution

(III) Classification

Def 2 (Definition): The order of PD is the order of highest derivative in PDE.

Examples:

① $u_t + u_x = 0$ 1st-order

② $u_{tt} - u_{xx} = f(x,t)$ 2nd-order

③ $u_t^5 + u_{xxxx} = 0$ 4-th order

Def 3: A PDE is linear if F in (*) is linear in u and its derivatives.

① linear, ② linear

③ nonlinear

④ $\Delta u = u^3$ nonlinear.

⑤ $x^2 u_x + e^{xy} u_y + \sin(x+y)\, u = x^2 y$

linear.

Def 4: If nonlinear term is only about u (not its derivatives) then it is semi-linear. (½)

If nonlinear term does not appear in highest derivative then it is quasi linear (¼)

④ semilinear

⑥ $\Delta u = |\nabla u|^2 u$

2nd-order, quasi-linear.

(IV) superposition principle

[for linear PDE]

$$L[u] = f(x,y)$$

$L[u]$ is differential operator

e.g: $L[u] \triangleq u_{xx} - u_{yy}$

property: $L[a_1u_1 + a_2u_2] = a_1L[u_1] + a_2L[u_2]$

- superposition principle

(i) homogeneous PDE

$$L[u] \triangleq u_{xx} - u_{yy} = 0$$

$$\begin{cases} (u_1)_{xx} - (u_1)_{yy} = 0 \\ (u_2)_{xx} - (u_2)_{yy} = 0 \end{cases}$$

$\Rightarrow (u_1+u_2)_{xx} - (u_1+u_2)_{yy} = (u_1)_{xx} - (u_1)_{yy} + (u_2)_{xx} - (u_2)_{yy} = 0$

$u_1 + u_2$ is a solution.

(ii) non-homogeneous PDE

$L[u] \triangleq u_{xx} - u_{yy} = f(x,y)$ (assign define)

suppose $f = f_1 + f_2$

$$\begin{cases} (u_1)_{xx} - (u_1)_{yy} = f_1 & (1) \\ (u_2)_{xx} - (u_2)_{yy} = f_2 & (2) \end{cases}$$

(1) + (2) $\Rightarrow (u_1+u_2)_{xx} - (u_1+u_2)_{yy} = f$

$u_1 + u_2$ is solution to $L[u] = f$.

RK (Remark): method to construct complex solutions.

e.g.: $f = \underbrace{x^2}_{f_1} + \underbrace{\sin(x+y)}_{f_2}$

To do list.

① download syllabus.

② 1, 2(a) Homework. (submitted to crowdmark)

Next: §1.4 – §1.5