Lecture4 §2.4 – §2.5

Friday, April 10, 2020 9:06 AM

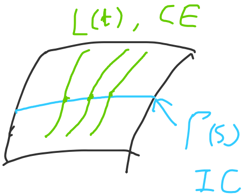
Objective $\begin{cases} \text{solving 1st-order PDE} \\ \text{existence \& uniqueness.} \end{cases}$

(I) Review

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) & \text{PDE} \\ (x_0(s), y_0(s), u_0(s)) \in \Gamma(s). & \text{IC} \end{cases}$$

- CE : $\begin{cases} x_t = a \\ y_t = b \\ u_t = c \end{cases}$

IC : $\begin{cases} x(0,s) = x_0(s) \\ y(0,s) = y_0(s) \\ u(0,s) = u_0(s). \end{cases}$



- Solve $\left.\begin{array}{l} x = x(t,s), \\ y = y(t,s) \end{array}\right\}$
  $u = u(t,s)$
- convert $u = u(x,y)$

$$(t,s) \leftrightarrow (x,y)$$

$$J \neq 0 \rightarrow \text{convertible.}$$

① $2u_x + 3u_y = 1$ PDE

→ $u(x,0) = e^x$ IC

$u(s,0) = e^s$

step 1:

CE: $\begin{cases} x_t = 2 & (1) \\ y_t = 3 & (2) \\ u_t = 1 & (3) \end{cases}$

IC: $\begin{cases} x(0,s) = s & (4) \\ y(0,s) = 0 & (5) \\ u(0,s) = e^s & (6) \end{cases}$

step 2:

(1) (4) $\Longrightarrow$

$$x(t,s) \overset{(1)}{=} \int_0^t 2\,dt + x(0,s)$$

$$\overset{(4)}{=} 2t + s \quad \leftarrow$$

(2) (5) $\Longrightarrow$

$$y(t,s) \overset{(2)}{=} \int_0^t 3\,dt + y(0,s)$$

$$\overset{(5)}{=} 3t + 0 = 3t \quad \leftarrow$$

(3) (6) $\Longrightarrow$

$$u(t,s) = \int_0^t 1\,dt + u(0,s)$$

$$= t + e^s$$

step 3: $u(x,y)$:

$$y = 3t \Rightarrow t = \frac{y}{3}$$

$$x = 2t + s \Rightarrow s = x - 2t = x - \frac{2}{3}y$$

$$u = t + e^{s} = \frac{y}{3} + e^{x - \frac{2}{3}y}$$

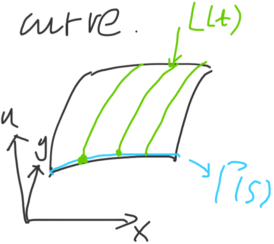
RK1: $J = \begin{vmatrix} x_t & y_t \\ x_s & y_s \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$

invertible → unique solution

RK2: characteristic curve. (L_p)

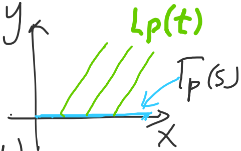
$\Gamma_p(s)$: $\begin{cases} x = s \\ y = 0 \end{cases}$

projection of $\Gamma(s)$



- $x - \frac{2}{3}y = s \rightarrow L_p$

fixed s



L_p: projection of $L(t)$

$$L_p: \frac{dy}{dx} = \frac{y_t}{x_t} = \frac{b}{a}$$

$$\text{in } ① \Rightarrow \frac{dy}{dx} = \frac{3}{2} \rightarrow L_p$$

(II) more examples.

②
$$\begin{cases} -y u_x + x u_x = u & \text{PDE} \\ u(x,0) = \psi(x) & \text{IC.} \\ \quad (x>0) & \end{cases}$$

step 1:

$$\text{CE}: \begin{cases} x_t = -y & (1) \\ y_t = x & (2) \\ u_t = u & (3) \end{cases}$$

$u(s,0) = \psi(s)$, $s>0$

$$\text{IC}: \begin{cases} x(0,s) = s & (4) \\ y(0,s) = 0 & (5) \\ u(0,s) = \psi(s), & (6) \end{cases}$$

step 2:

$$(1)(2) \Rightarrow x_{tt} \overset{(1)}{=} -y_t \overset{(2)}{=} -x$$

$$x_{tt} + x = 0$$

$$x(t,s) = \underline{C_1(s)\cos(t) + C_2(s)\sin(t)}$$

$$(1) \Rightarrow y = -x_t = C_1 \sin(t) - C_2 \cos(t)$$

$$(4,5) \Rightarrow \begin{cases} x(0,s) = C_1 + 0 = s \\ y(0,s) = 0 - C_2 = 0 \end{cases}$$

$\Rightarrow C_1 = s, \quad C_2 = 0$

$$\boxed{x = s\cdot\cos(t), \quad y = s\cdot\sin(t)}$$

$u_t = u$

(3) $\Rightarrow$ $u = \underline{C_3(s)e^{t}}$.

(6) $\Rightarrow$ $u(0,s) = C_3(s) \overset{(6)}{=} \psi(s)$

$$\boxed{u = \psi(s)\cdot e^{t}}$$

step3: $(t,s)\rightarrow(x,y)$

$$s = \sqrt{x^2+y^2}, \quad t = \arctan\left(\frac{y}{x}\right)$$

$$u = \psi\left(\sqrt{x^2+y^2}\right) e^{\arctan\left(\frac{y}{x}\right)} .$$

$$\begin{aligned} x^2+y^2 &= s^2(\cos t)^2 + s^2(\sin t)^2 \\ &= s^2[\cos^2 t + \sin^2 t] \\ &= s^2 \end{aligned}$$

$$\frac{y}{x} = \frac{\sin t}{\cos t} = \tan(t)$$

RK1:

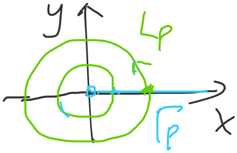
$$\begin{aligned} J &= \begin{vmatrix} x_t & y_t \\ x_s & y_s \end{vmatrix} \\ &= \begin{vmatrix} -s\cdot\sin(t), & s\cdot\cos t \\ \cos t & \sin t \end{vmatrix} \\ &= -s\sin^2 t - s\cos^2 t \end{aligned}$$

in ②, $= -s \neq 0$ if $s \neq 0$.

$s > 0 \Rightarrow J \neq 0$, invertible.

RK2:

$$\Gamma_p \begin{cases} x = s > 0 \\ y = 0 \end{cases}$$



L_p: $x^2+y^2=s^2$, fixed s

characteristic curve

③ Example 2.8.

$$\begin{cases} (y+u)\,u_x + y\,u_y = x-y \\ u(x,1) = 1+x. \end{cases}$$

$u(s,1) = 1+s$

step 1:

CE:
$$\begin{cases} x_t = y+u & (1) \\ y_t = y & (2) \\ u_t = x-y & (3) \end{cases}$$

IC:
$$\begin{cases} x(0,s) = s & (4) \\ y(0,s) = 1 & (5) \\ u(0,s) = 1+s & (6) \end{cases}$$

step 2:

$(2) \Rightarrow y = C_1(s)\cdot \underline{e^t}$

$(5) \Rightarrow y(0,s) = C_1(s) \overset{(5)}{=} 1$

$\Rightarrow \underline{y = e^t}$. (7)

$$(1)(3) \Rightarrow \begin{aligned} x_{tt} &\overset{(1)}{=} y_t + u_t \\ &\overset{(3)}{=} y_t + x - y \\ &\overset{(7)}{=} e^t + x - e^t \\ &= x \end{aligned}$$

$$\underline{x = C_2(s)\,e^t + C_3(s)\,e^{-t}}$$

$$(1) \Rightarrow \begin{aligned} u &= x_t - y \\ &= C_2 e^t - C_3 e^{-t} - e^t \end{aligned}$$

$$(4,6) \Rightarrow C_2 = 1+s, \quad C_3 = -1$$ [Exercise]

So: $$\boxed{\begin{aligned} x &= (1+s)e^t - e^{-t} \\ y &= e^t \\ u &= se^t + e^{-t} \end{aligned}}$$

RK: refer to textbook for another method.

step 3: $(s,t) \longleftrightarrow (x,y)$ [Exercise]

$$u = x - y + \frac{2}{y}$$

(III) Existence & uniqueness.

Def: transversality condition on $\Gamma(s)$ is

$$J\Big|_{\Gamma(s)} = J\Big|_{t=0} = \begin{vmatrix} x_t, & y_t \\ x_s, & y_s \end{vmatrix}\Bigg|_{t=0}$$

$$= \begin{vmatrix} a & b \\ (x_0)_s & (y_0)_s \end{vmatrix} \neq 0$$

Theorem:

$$\begin{cases} a u_x + b u_y = c \\ \Gamma(s) \end{cases}$$

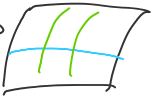
(1) $J \neq 0$ on $\Gamma(s)$

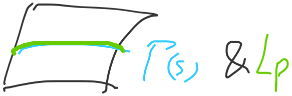
$\Rightarrow$ unique solution.

(2) $J = 0$. on part of $\Gamma(s)$.

$$\Rightarrow \begin{cases} \text{no solution} \\ \text{or } \infty \text{ many solutions} \end{cases}$$

RK:

$J \neq 0 \rightarrow$ 

$J = 0 \longrightarrow$ 

they coincide.

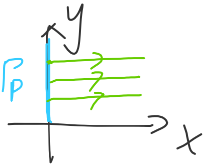
① recall:

$$\begin{cases} u_x = c_0 u + c_1 \\ u(0,y) = y \end{cases} \quad \rightarrow \quad u(0,s)=s$$

$x_0=0,\ y_0=s$

$$J = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0 .$$

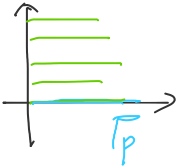
unique solution.



② $$\begin{cases} u_x = c_0 u + c_1 \\ u(x,0) = x \end{cases} \quad IC$$

$x_0=s,\ y_0=0$

$$J = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$



$\Rightarrow \begin{cases} \text{no solution} \\ \text{or } \infty \text{ many solutions} \end{cases}$

we showed $\underline{\text{no solution}}$

③ $$\begin{cases} u_x + u_y = 1 \\ u(x,x) = x \end{cases} \quad (IC)$$

discuss existence & uniqueness.

$a=1,\quad b=1$

$u(s,s)=s$

$x_0 = x(0,s) = s$

$y_0 = y(0,s) = s$

$$J = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \quad \text{on } \Gamma(s)$$

$$= 1 - 1 = 0$$

$\Rightarrow$ it has either no solution

or ∞ many solution.

[Exercise] no solution?

To do list.

① HW#4

② Exercises (chapter 2)

1-4, 6, 7, 11, 12, 16, 21

Next: chapter 5.