Lecture5

Wednesday, April 15, 2020 9:25 AM

§5.1 – §5.2

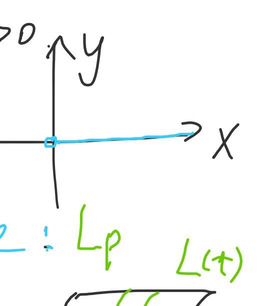
Objective { review. chapter 2.
method of separation of variables.

(1) Review.

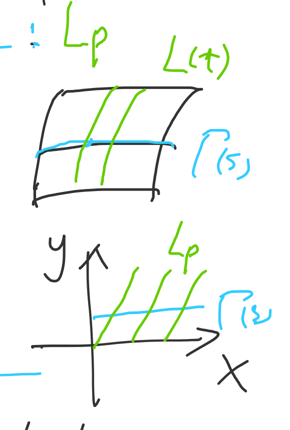
HW4 (b):

the entire ray $x>0$

$[x>0, y=0]$



- characteristic curve: L_p



① $2u_x + 3u_y = 1$

method 1:

$\Rightarrow x = 2t + s$
$y = 3t$

$\Rightarrow x = \frac{2y}{3} + s$ ←

L_p! (s is constant) → $y = \frac{3(x-s)}{2}$

method 2:

$a(x,y)u_x + b(x,y)u_y = c$

L_p: $\frac{dy}{dx} = \frac{y_t}{x_t} = \frac{b(x,y)}{a(x,y)}$

solve $y(x)$

For ①: $\frac{dy}{dx} = \frac{3}{2} \Rightarrow y = \frac{3}{2}x + C$ ←

RK: $C = \frac{-3}{2}s$

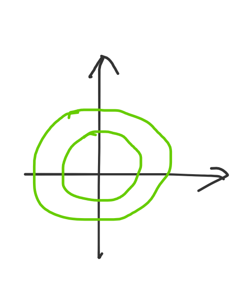
② $-yu_x + xu_y = u$. find L_p.

$\frac{dy}{dx} = \frac{x}{-y}$

$-y\,dy = x\,dx$

$-\frac{y^2}{2} = \frac{x^2}{2} + C$

$x^2 + y^2 = -2C = C_1$



Exercise: find L_p for $u_x + y^{2/3}u_y = 0$.

- Existence & uniqueness.

③ $\begin{cases} u_x + u_y = 1 & \text{(PDE)} \\ u(x,x) = x & \text{(IC)} \end{cases}$

step 1: $J = 0 \Rightarrow \begin{cases} \text{no solution} \\ \text{or } \infty \text{ many solutions} \end{cases}$

step 2: → $u(x,y) = x$ check PDE & IC

It is a solution.

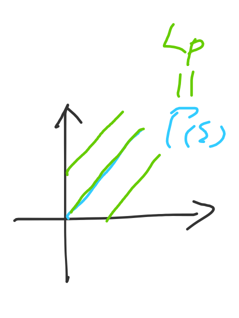
$\Rightarrow \infty$ many solutions.

RK: $u(x,y) = \frac{x+y}{2}$ is also a solution

④ $\begin{cases} u_x + u_y = 1 & \text{(PDE)} \\ u(x,x) = 1 \end{cases}$

step 1: same as ③.

step 2: $u = 1$ ✗



along Γ(s): $\begin{cases} x = s \\ y = s \\ u = 1 \end{cases}$ (*)

(i) → $\frac{du}{ds} = u_x x_s + u_y y_s$
$= u_x + u_y \overset{\text{PDE}}{=} 1$

(ii) (*) $u = 1 \overset{\text{IC}}{\Rightarrow} \frac{du}{ds} = 0$ contradiction.

$\Rightarrow$ no solution.

(II): chapter 3 & 4.

ch3: 2nd-order PDE (linear)

$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$

$\begin{cases} \delta = b^2 - ac > 0, \text{ hyperbolic, eg: } u_{tt} = u_{xx} \\ \delta = 0 \quad \text{parabolic, eg: } u_t = u_{xx}. \\ \delta < 0, \text{ elliptic, e.g. } u_{xx} + u_{yy} = 0. \end{cases}$

ch4:

$u_{tt} = c^2 u_{xx} + g \qquad x \in (-\infty, \infty)$

→ d'Alembert's formula.

(III) method of separation of variable.

heat equation

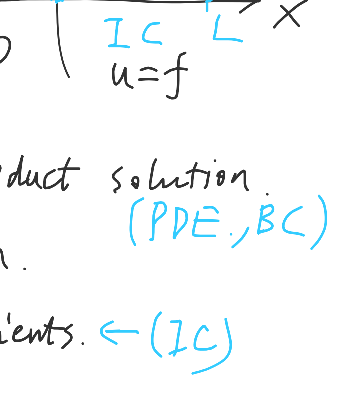
$u_t = \nabla \cdot (k \nabla u) + q$ Fourier.

Case: k = constant, $q = 0$, 1D space.

$(*) \begin{cases} u_t = ku_{xx}, \ 0<x<L, \ t>0. & \text{(PDE)} \\ u(0,t) = u(L,t) = 0, \ t \geq 0. & \text{(BC)} \\ u(x,0) = f(x), \ 0 \leq x \leq L & \text{(IC)} \end{cases}$

RK: compatibility condition

$f(0) = 0, f(L) = 0$



method:

step 1: separated/product solution (PDE, BC)

step 2: superposition.

step 3: find coefficients. ← (IC)

step 1: (1a)

we seek solution of the form.

$u(x,t) = X(x) \cdot T(t)$

PDE $\Rightarrow u_t = ku_{xx}$

$\Rightarrow X \cdot T_t = kX_{xx}T$

$\Rightarrow \frac{T_t}{kT} = \frac{X_{xx}}{X} = -\lambda$

$\Rightarrow \begin{cases} X_{xx} = -\lambda X & (1) \\ T_t = -k\lambda T. & (2) \end{cases}$

BC $\Rightarrow u(0,t) = X(0) \cdot T(t) = 0$
$u(L,t) = X(L)\,T(t) = 0$

$\xrightarrow{T \neq 0} X(0) = X(L) = 0$. (3)

(1b) Solve $X(x)$

$\begin{cases} X_{xx} + \lambda X = 0 & (1) \\ X(0) = X(L) = 0 & (3). \end{cases}$

case 1: $\lambda < 0$.

$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{\sqrt{-\lambda}x}$

$\begin{cases} X(0) = C_1 + C_2 = 0 \\ X(L) = C_1 e^{\sqrt{-\lambda}L} + C_2 e^{\sqrt{-\lambda}L} = 0 \end{cases}$

$\Rightarrow C_1 = C_2 = 0$.

case 2: $\lambda = 0$

(1) $\Rightarrow X_{xx} = 0$

$X(x) = C_1 + C_2 x$.

(3) $\Rightarrow \begin{cases} X(0) = C_1 = 0 \\ X(L) = C_1 + C_2 \cdot L = 0 \end{cases}$

$C_1 = C_2 = 0$.

case 3: $\lambda > 0$

(1) $\Rightarrow X(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$

(3) $\Rightarrow \begin{cases} X(0) = C_1 = 0 \\ X(L) = 0 + C_2\sin(\sqrt{\lambda}L) = 0 \end{cases}$

$C_2 \neq 0, \ C_2 = 1$ for nontrivial solution

$\sin(\sqrt{\lambda}L) = 0$.

$\sqrt{\lambda}L = n\pi, \quad n = 1, 2, 3, \cdots$

$\begin{cases} \lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2 \\ X = X_n = \sin\left(\frac{n\pi}{L}x\right), \ n = 1, 2, 3, \cdots \end{cases}$

(1c): Solve $T(t)$

$T_t = -k\lambda T$.

$\Rightarrow T = Be^{-k\lambda t}$

substitute $\lambda = \lambda_n$.

$T = T_n = B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

summary:

$u_n = X_n \cdot T_n$
$= B_n \cdot \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

$n = 1, 2, 3, \cdots$

step 2: superposition [formal]

$u(x,t) = \sum_{n=1}^{\infty} u_n$
$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$.

step 3: Find B_n.

IC $\Rightarrow u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$

B_n: Fourier coefficient.

Fourier expasion of $f(x)$

multiply $\sin\left(\frac{m\pi x}{L}\right)$ & integrate: [fixed m]

$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$
$= \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx$

Formula $\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n \geq 1 \end{cases}$

↑ [Exercise]

$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$
$= B_m \cdot \frac{L}{2}$

$\Rightarrow B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx$.

$m = 1, 2, \cdots$

Summary:

$u(x,t) = \sum_{m=1}^{\infty} B_m \sin\left(\frac{m\pi x}{L}\right) e^{-k\left(\frac{m\pi}{L}\right)^2 t}$

RK: $f(x)$ is assumed to be piecewise continuous.

To do list.

① Exercise.

② HW5.

Next: §5.2 – §5.3