Lecture6 §5.2 – §5.3

Friday, April 17, 2020 9:23 AM

Objective: { heat equation (review); other BC; wave equation

(I) Review

$$\begin{cases} u_t = k u_{xx}, & 0<x<L,\ t>0 \quad \text{(PDE)} \\ u(0,t)=u(L,t)=0, & t\ge 0 \quad \text{(BC)} \\ u(x,0)=f(x), & \quad \text{(IC)} \end{cases}$$

Solution:

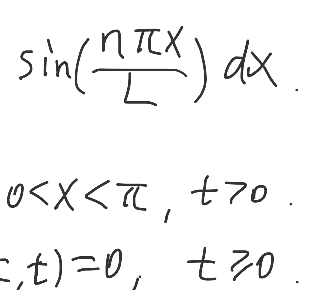
$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Example:

$$\begin{cases} u_t = u_{xx}, & 0<x<\pi,\ t>0. \\ u(0,t)=u(\pi,t)=0, & t\ge 0. \\ u(x,0)=f(x)=\begin{cases} x, & 0\le x\le \frac{\pi}{2} \\ \pi-x, & \frac{\pi}{2}\le x\le \pi \end{cases} \end{cases}$$

$\boxed{\frac{n\pi}{L}=n}$



Solution: $k=1$, $L=\pi$

$$u=\sum_{n=1}^{\infty} B_n \sin(nx)\cdot e^{-n^2 t}$$

$$B_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\sin(nx)dx.$$

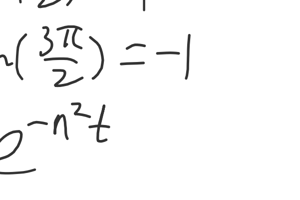
By integration by parts

$$=\frac{2}{\pi}\left[-\frac{x}{n}\cos(nx)+\frac{\sin(nx)}{n^2}\right]\Big|_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left[\frac{(x-\pi)}{n}\cos(nx)-\frac{\sin(nx)}{n^2}\right]\Big|_{\frac{\pi}{2}}^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{-\pi}{2n}\cos\left(n\frac{\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\cdot\frac{1}{n^2} - 0\right] + \frac{2}{\pi}\left[0+\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)+\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right]$$

$$=\frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right),\quad n\ge 1.$$

If $n=2m$, $m\ge 1$.



$$\sin\left(\frac{n\pi}{2}\right)=\sin(m\pi) = 0$$

If $n=2m-1$, $m\ge 1$

$$\sin\left(\frac{n\pi}{2}\right)=\sin\left(m\pi-\frac{\pi}{2}\right)=(-1)^{m+1}$$

$m=1$, $n=1$. $\sin(\frac{\pi}{2})=1$

$m=2$, $n=3$, $\sin(\frac{3\pi}{2})=-1$

$$u=\sum_{n=1}^{\infty}B_n\sin(nx)e^{-n^2t}$$

$$\sum_{n=1}^{\infty}=\sum_{n=2m}+\sum_{n=2m-1}$$

($B_n=0$) → 0

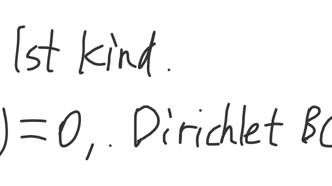
$$=\sum_{\substack{m=1\\ n=2m-1}}^{\infty} B_n\sin(nx)e^{-n^2t}$$

$$=\sum_{\substack{m=1\\ n=2m-1}}^{\infty}\frac{4\sin\left(\frac{n\pi}{2}\right)}{\pi n^2}\sin(nx)e^{-n^2t}$$

$$=\sum_{m=1}^{\infty}\frac{4\cdot(-1)^{m+1}}{\pi(2m-1)^2}\sin((2m-1)x)\cdot e^{-(2m-1)^2t}.$$

RK1: It converges absolutely.

RK2: $t\to\infty$, $u\to 0$.



RK3: $f(x)$ is not smooth → u at t=0

· $t>0$. u is smooth. smoothing effect of heat equation.

(II). other types of BC.

① In (I): BC of 1st kind.

$u(0,t)=u(L,t)=0$, Dirichlet BC.

$X_n=\sin\left(\frac{n\pi x}{L}\right)$, $n=1,2,\cdots$

② BC of 2nd kind [Neumann BC]

$u_x(0,t)=u_x(L,t)=0$.

$X_n=\cos\left(\frac{n\pi x}{L}\right)$, $n=0,1,2,\cdots$

③ BC of 3rd kind [Robin BC].

$a u_x(0,t)+b u(0,t)=0$.

$c u_x(L,t)+d u(L,t)=0$.

$a=c=0$, $b=d=1$, ⇒ ①

$a=c=1$, $b=d=0$ ⇒ ②

$a=b=c=d=1$.

④ mixed BC. e.g. $\begin{cases} u(0,t)=0 \\ u_x(L,t)=0 \end{cases}$

⑤ periodic BC. ⋯⋯ (see textbook)

RK: Find X_n for ③ ④.

(III) wave equation.

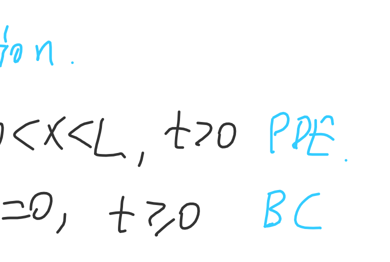
$$\begin{cases} u_{tt}=c^2u_{xx}, & 0<x<L,\ t>0 \quad \text{PDE.} \\ u_x(0,t)=u_x(L,t)=0, & t\ge 0 \quad \text{BC} \\ u(x,0)=f(x), & 0\le x\le L, \quad (IC)_1 \\ u_t(x,0)=g(x), & 0\le x\le L, \quad (IC)_2 \end{cases}$$

RK1 c: wave speed.

RK2: compatibility condition

$f'(0)=f'(L)=0$

$g'(0)=g'(L)=0$.



Step1 (1a). separated solution

$u=X(x)T(t)$

PDE ⇒ $u_{tt}=c^2u_{xx}$

$XT_{tt}=c^2X_{xx}T$

$$\frac{T_{tt}}{c^2T}=\frac{X_{xx}}{X}=-\lambda$$

$$\begin{cases} X_{xx}=-\lambda X & (1) \\ T_{tt}=-\lambda c^2 T & (2). \end{cases}$$

BC ⇒ $X_x(0)T(t)=0$.

$X_x(L)T(t)=0$.

$\xrightarrow{T\neq 0}$ $X_x(0)=X_x(L)=0.$ (3)

(1b). Solve $X(x)$

$$\begin{cases} X_{xx}=-\lambda X, & (1) \\ X_x(0)=X_x(L)=0, & (3) \end{cases}$$

Case1: $\lambda<0$. [trivial solution Exercise]

Case2: $\lambda=0$.

$X(x)=C_1+C_2x$, $X_x=C_2$

$$\begin{cases} X_x(0)=C_2=0 \\ X_x(L)=C_2=0 \end{cases}$$

Choose $C_1=1$, $X(x)=1$.

Case3: $\lambda>0$.

(1) ⇒ $X(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$.

$X_x=-C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x)+C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

(3) ⇒ $X_x(0)=0+\sqrt{\lambda}C_2=0$

$C_2=0$.

$X_x(L)=-C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L)+0 = 0$.

non-trivial solution: $C_1\neq 0$, $C_1=1$.

$\sin(\sqrt{\lambda}L)=0$.

$\sqrt{\lambda}L=n\pi$, $n=1,2,3,\cdots$ ($\lambda>0$)

⇒ $\lambda=\lambda_n=\left(\frac{n\pi}{L}\right)^2$, $n=1,2,3,\cdots$

$X=X_n=\cos\left(\frac{n\pi}{L}x\right)$.

$n=1,2,3,\cdots$

Summary of (1b)

$X_n=\cos\left(\frac{n\pi}{L}x\right)$

$\lambda_n=\left(\frac{n\pi}{L}\right)^2$, $n=0,1,2,\cdots$

RK: $n=0$, → $\lambda_0=0$,

$X_0=1$ → case 2.

(1c) Solve $T(t)$

$T_{tt}=-\lambda c^2 T$.

· $\lambda=\lambda_0=0$, ⇒ $T_{tt}=0$.

$$T(t)=T_0=\frac{A_0+B_0t}{2}$$

· $\lambda=\lambda_n=\left(\frac{n\pi}{L}\right)^2$, $n=1,2,3,\cdots$

$\sqrt{\lambda c^2}=\frac{n\pi c}{L}$

$$\Rightarrow T_n=A_n\cos\left(\frac{n\pi c}{L}t\right)+B_n\sin\left(\frac{n\pi c}{L}t\right).$$

Step2: superposition.

$u_n=X_nT_n$.

$$u(x,t)=\sum_{n=0}^{\infty}u_n = \frac{A_0+B_0t}{2} + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi c}{L}t\right)+B_n\sin\left(\frac{n\pi c}{L}t\right)\right]$$

Step3: Find A_n, B_n, $n=0,1,2,\cdots$

$(IC)_1$ ⇒ $f(x)=u(x,0)$

$$=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)$$

$(IC)_2$ ⇒ $g(x)=u_t(x,0)$

$$=\frac{B_0}{2}+\sum_{n=1}^{\infty}B_n\frac{n\pi c}{L}\cos\left(\frac{n\pi x}{L}\right).$$

[Exercise]

To do list.

① HW6.

② Exercises for chapter 5.

Next: §5.3 – §5.4.