Lecture7

Wednesday, April 22, 2020 9:25 AM

§5.3 – §5.4

Objective { wave equation; non-homogeneous Equations.

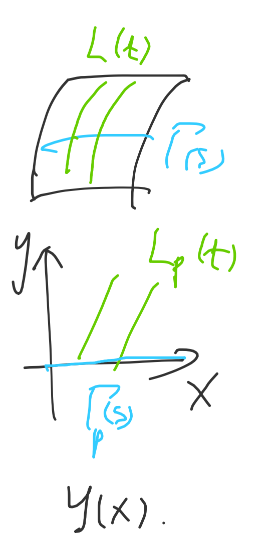
(I) Review.

Clarification:

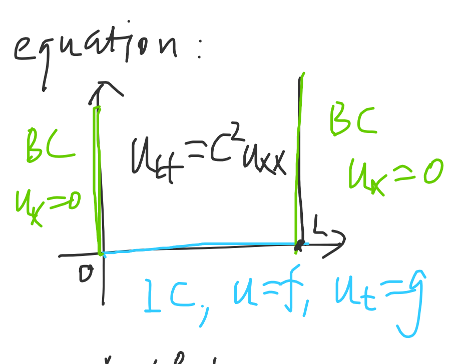
$L(t)$: characteristic curves.

L_p: projection of $L(t)$ on x-y plane.

L_p: characteristics



wave equation:



$$u=\sum_{n=0}^{\infty}u_n=\frac{A_0+B_0t}{2}+\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi c}{L}t\right)+B_n\sin\left(\frac{n\pi c}{L}t\right)\right]$$

step 3:

$(IC)_1 \Rightarrow f(x)=u(x,0)$
$$=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right) \quad (*)_1$$

$(IC)_2 \Rightarrow g(x)=u_t(x,0)$
$$=\frac{B_0}{2}+\sum_{n=1}^{\infty}B_n\frac{n\pi c}{L}\cos\left(\frac{n\pi x}{L}\right) \quad (*)_2$$

formula: $\int_0^L\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi x}{L}\right)dx$
$$=\begin{cases}0, & \text{if } m\neq n\\ L/2 & \text{if } m=n\neq 0\\ L & \text{if } m=n=0\end{cases}$$

multiply $\cos\left(\frac{m\pi x}{L}\right)$ & integrate.

$(*)_1 \Rightarrow A_m=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx$

$m=0,1,2,\cdots$

$(*)_2 \Rightarrow B_m=\begin{cases}\frac{2}{L}\int_0^L g(x)\,dx & \text{if } m=0\\ \frac{L}{c\pi n}\frac{2}{L}\int_0^L g(x)\cos\left(\frac{m\pi x}{L}\right)dx & \text{if } m=1,2,\cdots\end{cases}$

RK: In heat equation, e^{-c_nt} (parabolic) smoothing effect.

In wave Equation, $\sin(c_nt)$, $\cos(c_nt)$. (hyperbolic) no smoothing effect. It preserves singularity.

(II) Example.

$$\begin{cases}u_{tt}=4u_{xx}, & 0<x<1,\ t>0.\\ u_x(0,t)=u_x(1,t)=0, & t\geq 0.\\ u(x,0)=f(x)=\cos^2(\pi x), & 0\leq x\leq 1 \quad (IC)_1\\ u_t(x,0)=g(x)=\sin^2(\pi x)\cos(\pi x) & (IC)_2 \quad 0\leq x\leq 1\end{cases}$$

Solution: $c^2=4$, $c=2$

$L=1$

$$u(x,t)=\frac{A_0+B_0t}{2}+\sum_{n=1}^{\infty}\cos(n\pi x)\cdot\left[A_n\cos(2n\pi t)+B_n\sin(2n\pi t)\right]$$

$$u(x,0)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos(n\pi x)$$
$\overset{(IC)_1}{=} f(x)=\cos^2(\pi x)$
$$=\frac{1+\cos(2\pi x)}{2}$$
$$=\frac{1}{2}+\frac{1}{2}\cos(2\pi x).$$

compare coefficient of $\cos(n\pi x)$.

$n=0$, $\frac{A_0}{2}=\frac{1}{2} \Rightarrow A_0=1$.

$n=2$, $A_2=\frac{1}{2}$

$A_n=0$, if $n\neq 0,2$.

$g(x)=\sin^2(\pi x)\cos(\pi x)$.
$$=\frac{1}{2}[1-\cos(2\pi x)]\cdot\cos(\pi x)$$
$$=\frac{1}{2}\cos(\pi x)-\cos(2\pi x)\cos(\pi x)$$
$$=\frac{1}{2}\cos(\pi x)-\frac{1}{4}[\cos(3\pi x)+\cos(\pi x)]$$
$$=\frac{1}{4}\cos(\pi x)-\frac{1}{4}\cos(3\pi x)$$

RK: $\cos(a)\cos(b)=\frac{1}{2}[\cos(a+b)+\cos(a-b)]$

$$u_t(x,0)=\frac{B_0}{2}+\sum_{n=1}^{\infty}B_n 2n\pi\cos(n\pi x)$$
$\overset{(IC)_2}{=} g$

$\Rightarrow n=1$, $B_1\cdot 2\pi=\frac{1}{4}$
$B_1=\frac{1}{8\pi}$

$n=3$: $B_3\cdot 6\pi=-\frac{1}{4}$
$B_3=\frac{-1}{24\pi}$

$B_n=0$, if $n\neq 1,3$

$$u(x,t)=\frac{1}{2}+\frac{1}{8\pi}\cos(\pi x)\sin(2\pi t)+\frac{1}{2}\cos(2\pi x)\cos(4\pi t)-\frac{1}{24\pi}\cos(3\pi x)\sin(6\pi t).$$

RK: use formula to compute A_n, B_n [Exercise]

(III) Non-homogeneous Equations.

$$\begin{cases}u_{tt}-c^2u_{xx}=F(x,t), & 0<x<L,\ t>0.\\ u_x(0,t)=u_x(L,t)=0. & t\geq 0.\\ u(x,0)=f,\ u_t(x,0)=g, & 0\leq x\leq L \quad (IC)\end{cases}$$

step 1: $u_{tt}=c^2u_{xx}$. $[F=0]$

$X_n=\cos\left(\frac{n\pi x}{L}\right)$

$n=0,1,2,\cdots$

[use X_n in this system]

step 2: $F(x,t)=\sum_{n=0}^{\infty}F_n(t)\cos\left(\frac{n\pi x}{L}\right)$

$$u=\sum_{n=0}^{\infty}X_nT_n=\sum_{n=0}^{\infty}\cos\left(\frac{n\pi x}{L}\right)T_n(t).$$

PDE $\Rightarrow$
$$\sum_{n=0}^{\infty}\cos\left(\frac{n\pi x}{L}\right)T_n''(t)-c^2\sum_{n=0}^{\infty}(-1)\cdot\left(\frac{n\pi}{L}\right)^2\cos\left(\frac{n\pi x}{L}\right)T_n=\sum_{n=0}^{\infty}F_n\cos\left(\frac{n\pi x}{L}\right)$$

compare coefficient of $\cos\left(\frac{n\pi x}{L}\right)$.

$$T_n''(t)+c^2\left(\frac{n\pi}{L}\right)^2T_n=F_n(t)$$

Solve $T_n(t)$, with integrating constants A_n, B_n.

step 3: Determine A_n & B_n.

IC

Example:

$$\begin{cases}u_{tt}-u_{xx}=\cos(2\pi x)\sin(2\pi t), & 0<x<1\\ u_x(0,t)=u_x(1,t)=0, & t>0.\\ u(x,0)=f(x)=\cos^2(\pi x), & 0\leq x\leq 1 \quad (IC)_1\\ u_t(x,0)=g(x)=2\cos(2\pi x), & 0\leq x\leq 1 \quad (IC)_2\end{cases}$$

$L=1$

step 1: $X_n=\cos(n\pi x)$, $n=0,1,2,\cdots$

step 2: $u(x,t)=\sum_{n=0}^{\infty}T_n(t)\cos(n\pi x)$

PDE $\Rightarrow$
$$\sum_{n=0}^{\infty}T_n''(t)\cos(n\pi x)+\sum_{n=0}^{\infty}n^2\pi^2T_n\cos(n\pi x)=\cos(2\pi x)\cos(2\pi t)$$

$$\begin{cases}T_0''(t)=0\\ T_2''(t)+4\pi^2T_2=\cos(2\pi t).\\ T_n''+n^2\pi^2T_n=0, & \text{if } n\neq 0,2\end{cases}$$

$$T_0=\frac{A_0+B_0t}{2}$$

$$T_n=A_n\cos(n\pi t)+B_n\sin(n\pi t).$$

$$T_2=A_2\cos(2\pi t)+B_2\sin(2\pi t)+\frac{t}{4\pi}\sin(2\pi t).$$

$$u(x,t)=\frac{A_0+B_0t}{2}+\frac{t}{4\pi}\sin(2\pi t)\cos(2\pi x)+\sum_{n=1}^{\infty}\left[A_n\cos(n\pi t)+B_n\sin(n\pi t)\right]\cdot\cos(n\pi x) \quad (*)$$

step 3:

$(IC)_1 \Rightarrow u=f=\cos^2\pi x$
$$=\frac{1+\cos(2\pi x)}{2}$$

$(*) \Rightarrow u(x,0)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos(n\pi x)$.

$A_0=1$, $A_2=\frac{1}{2}$

$A_n=0$, if $n\neq 0,2$,

$$(*) \Rightarrow u_t(x,t)=\frac{B_0}{2}+\frac{\cos(2\pi x)}{4\pi}\left[\sin(2\pi t)+t\cos(2\pi t)\cdot 2\pi\right]+\sum_{n=1}^{\infty}\left(-A_nn\pi\sin(n\pi t)+B_nn\pi\cos(n\pi t)\right)\cdot\cos(n\pi x)$$

$$u_t(x,0)=\frac{B_0}{2}+\sum_{n=1}^{\infty}B_n\,n\pi\cos(n\pi x).$$
$\overset{(IC)_2}{=} g=2\cos(2\pi x)$.

$$\Rightarrow\begin{cases}B_2\cdot 2\pi=2 \Rightarrow B_2=\frac{1}{\pi}\\ B_n=0, & \text{if } n\neq 2\end{cases}$$

Final solution:
$$u(x,t)=\frac{1}{2}+\left[\frac{1}{2}\cos(2\pi t)+\frac{t+4}{4\pi}\sin(2\pi t)\right]\cdot\cos(2\pi x)$$

RK: In heat Equation, wave equation we set $x\in[0,L]$

$\Rightarrow X_n \to$ "sin" "cos"

If $x\in[0,\infty)$, $X_n\to e^{-\lambda_n x}$

[see §5.6.2]

To do list

HW7.

Next: §5.5