Lecture8 §5.5

Friday, April 24, 2020 9:27 AM

Objective { uniqueness by energy method
Review
Fourier coefficients.

(I) Energy method.

$$\begin{cases} u_t - Ku_{xx} = F(x,t), 0<x<L, t>0 \\ u(0,t)=a(t),\ u(L,t)=b(t),\ t\geq 0 \quad \text{BC} \\ u(x,0)=f(x),\ 0\leq x\leq l \end{cases}$$

Aim: prove uniqueness.

Let u_1, u_2 be two solutions

define $w = u_1 - u_2$.

$w=0 \Leftrightarrow u_1 = u_2$

PDE $\Rightarrow$ $(u_1)_t - K(u_1)_{xx} = F$ (1)

$(u_2)_t - K(u_2)_{xx} = F$ (2)

(1)−(2): $w_t - k w_{xx} = 0$

BC $\Rightarrow$ $w(0,t) = w(L,t) = 0$

(IC) $\Rightarrow$ $w(x,0) = 0$

uniqueness $\Leftrightarrow$ $w=0$.

Define energy:

$$E(t) = \frac{1}{2}\int_0^L w^2(x,t)\,dx \geq 0$$

$$E'(t) = \frac{1}{2}\int_0^L 2w\, w_t\, dx$$

$= \int_0^L w\, w_t\, dx$

$\overset{PDE}{=} \int_0^L w\, K w_{xx}\, dx$

$= k\int_0^L w\, d(w_x)$

$= k\, w w_x \Big|_0^L - k\int w_x w_x dx$

$\overset{BC}{=} 0 - k\int_0^L w_x^2\, dx$

≤ 0

$E(0) = \frac{1}{2}\int_0^L w^2(x,0)\,dx = 0$

$E(t) \leq E(0) = 0$ $t>0$

$\Rightarrow E(t) = 0$ for all t

$\frac{1}{2}\int_0^L w^2 dx = 0$.

$\Rightarrow w = 0$

RK: For wave Equation.

$w = u_1 - u_2$

$$\begin{cases} w_{tt} = c^2 w_{xx} \\ BC,\ IC. \end{cases}$$

$$E = \frac{1}{2}\int_0^L w_t^2 + c^2 w_x^2\, dx$$

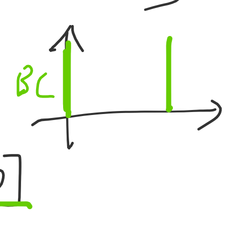
w_t^2: kinetic energy; $c^2 w_x^2$: elastic energy.

[Exercise]

(II) Review.

(A) Heat Equation. (homogeneous)

$$\begin{cases} u_t - Ku_{xx} = 0 \quad [u_t = Ku_{xx}] \\ BC \\ IC \end{cases}$$



(i) Dirichlet BC $[u=0]$

$\rightarrow X_n = \sin\left(\frac{n\pi x}{L}\right)$. $n=1,2,3,\cdots$

$\rightarrow$ solution $u = \sum T_n X_n = \cdots$

[Lecture 5]

(ii) Neumann BC $[u_x = 0]$

$\rightarrow X_n = \cos\left(\frac{n\pi x}{L}\right)$, $n=0,1,2,\cdots$

$\rightarrow u = \sum T_n X_n$

(iii) other BC ... [HW6]

(B) wave Equation.

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad [u_{tt} = c^2 u_{xx}] \\ BC \leftarrow \\ (IC)_1, (IC)_2 \end{cases}$$

(i) Dirichlet BC $[u=0]$

$\rightarrow X_n = \sin\left(\frac{n\pi x}{L}\right)$, $n=1,2,\cdots$

(ii) Neumann BC $[u_x=0]$

$\rightarrow X_n = \cos\left(\frac{n\pi x}{L}\right)$, $n=0,1,2,\cdots$

[Lecture 6]

Then $u = \sum X_n T_n$

(C) Non-homogeneous wave Equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = F \\ BC \\ (IC)_1, (IC)_2 \end{cases}$$

step1: use X_n from homogeneous Equation.

(i) Dirichlet BC $[u=0]$

$\rightarrow X_n = \sin\left(\frac{n\pi x}{L}\right)$

(ii) Neumann BC $[u_x=0]$

$\rightarrow X_n = \cos\left(\frac{n\pi x}{L}\right)$.

step2: $u = \sum T_n X_n$ no explicit formulas

$F = \sum F_n X_n$

$T_n'' + C_n T_n = F_n$

solve T_n.

step3: Determine A_n B_n by $(IC)_{1,2}$

(D) Nonhomogeneous heat Equation

$$\begin{cases} u_t - Ku_{xx} = F \\ BC \\ IC \end{cases}$$

step1: use X_n from homogeneous Equation.

(i) Dirichlet BC $[u=0]$

$\rightarrow X_n = \sin\left(\frac{n\pi x}{L}\right)$ ←

(ii) Neumann BC $[u_x=0]$

$\rightarrow X_n = \cos\left(\frac{n\pi x}{L}\right)$ ←

Step2: $u = \sum T_n X_n$.

$F = \sum F_n X_n$

$\rightarrow T_n' + C_n T_n = F_n$.

solve T_n

step3: Find A_n, B_n by (IC).

RK: see Exercise in textbook

(III) Fourier coefficient.

For $f(x)$ $x\in[0,L]$

(a) Fourier series 1 (half period)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

(b) Fourier series 2 (half period)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

RK1: (a) (b) correspond to X_n in (i) & (ii).

RK2: $\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx$

$= \int_0^L \frac{1-\cos\left(\frac{2n\pi x}{L}\right)}{2}dx$

$= \frac{1}{2}x - \frac{1}{2}\sin\left(\frac{2n\pi x}{L}\right)\frac{L}{2n\pi}\Big|_0^L$

$= \frac{L}{2}$

• For $f(x)$, $x\in[0,2L]$ (or $[-L,L]$)

Fourier series 3. (full period)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$$

where:

$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$n=0,1,2,\cdots$

$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

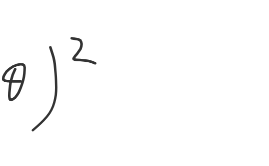
$n=1,2,\cdots$

RK: For periodic BC.

$\rightarrow X_n$: sin( ) & cos( )

(IV) Euler formula.

$e^{i\theta} = \cos(\theta) + i\sin\theta$.



derive trig identities

$e^{i2\theta} = \cos(2\theta) + i\sin(2\theta)$.

$e^{i2\theta} = (e^{i\theta})^2$

$= (\cos\theta + i\sin\theta)^2$

$= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= \cos^2\theta + \sin^2\theta - 2\sin^2\theta$

$= 1 - 2\sin^2\theta$.

$\sin(2\theta) = 2\sin\theta\cos\theta$.

RK: Derive $\cos(a)\cos(b)$

$= \frac{1}{2}[\cos(a+b) + \cos(a-b)]$

To do list.

HW8

Next: chapter 7.