Lecture9 ch 6 & ch 7

Wednesday, April 29, 2020 9:28 AM

Objective { review & ch 6
ch 7. [introduction]
Questions.

(I) Review.

recall: $u_t - ku_{xx} = F$, $[F=0]$

$u_{tt} - c^2 u_{xx} = F$, $[F=0]$

BC $\dashrightarrow X_n$

Sturm-Liouville problem.

$\{X_n, \lambda_n\}$

$\begin{cases} X'' = -\lambda X \\ X(0) = X(L) = 0 \end{cases}$ ← BC $\rightarrow X_n = \sin\left(\frac{n\pi x}{L}\right)$ $n = 1, 2, 3, \cdots$

$[u_x = 0]$ $\begin{cases} X'' = -\lambda X \\ X'(0) = X'(L) = 0 \end{cases}$ $\rightarrow X_n = \cos\left(\frac{n\pi x}{L}\right)$ $n = 0, 1, 2, \cdots$

periodic BC $\begin{cases} X'' = -\lambda X \\ X(0) = X(L) \\ X'(0) = X'(L) \end{cases}$ [HW6] $X_n = \cdots$ [derivation in chapter 6]

$\int_0^L X_n X_m \, dx = 0$, if $m \neq n$
orthogonality.

$\{X_n\}$ is complete orthogonal basis.

RK: Example 6.45 [textbook]
exercise of nonhomogeneous wave Equation

(II) Elliptic Equation (ch 7)

- Laplace Equation.

in 2-D

$\Delta u = 0 \iff u_{xx} + u_{yy} = 0$

function satisfying $\Delta u = 0$ is called harmonic function.

- $\Delta u = F$ Poisson's Eq.

$u_{xx} + u_{yy} = F$

Physics:

① heat Eq.

$u_t = k\Delta u + q$

at steady state: $u_t = 0$

$k\Delta u + q = 0$

$\Delta u = \frac{-q}{k}$ ← F

② Maxwell's Eq.

$\nabla \cdot (\vec{E}) = \frac{\rho}{\varepsilon_0}$. (1)

ρ: charge density.

$\vec{E}$: electric field.

$\vec{E} = -\nabla\phi$. (2)

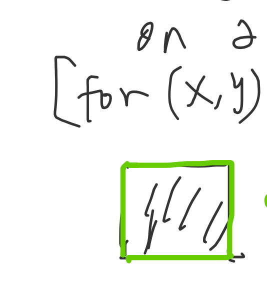
$\Rightarrow -\nabla \cdot (\nabla\phi) = \frac{\rho}{\varepsilon_0}$

$\Delta\phi = \frac{-\rho}{\varepsilon_0}$ ← F

(ϕ ↑ u)

BC [Boundary condition]

2D: $u_{xx} + u_{yy} = F$, | real 1D $u_{xx} = f$

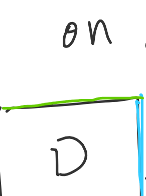


$(x, y) \in D$

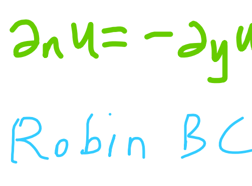
(i) Dirichlet BC

$u(x, y) = g(x, y)$ on ∂D.
[for $(x, y) \in \partial D$]

| $u(0) = u(L) = 0$
$\left[\begin{matrix} u(0) = a \\ u(L) = b \end{matrix}\right]$

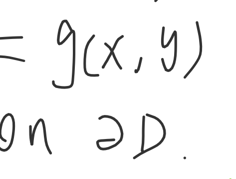


(2) Neumann BC



$\partial_n u(x, y) = g(x, y)$ on ∂D

| $u_x(0) = u_x(L) = 0$
$\left[\begin{matrix} u_x(0) = a \\ u_x(L) = b \end{matrix}\right]$



$\partial_n u = \partial_x u = u_x$

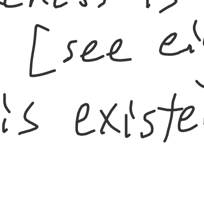
$\partial_n u = -\partial_y u = u_y$

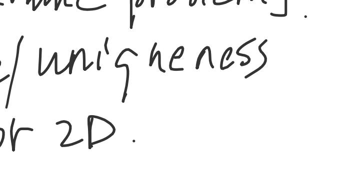
(3) Robin BC.

$u(x,y) + a(x,y)\partial_n u(x,y) = g(x, y)$ on ∂D.

| 1D
$u(0) + a u_x(0) = b$
$u(L) + c u_x(L) = d$

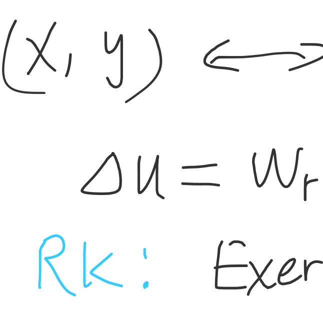
(4) mixed.





RK: for BVP. (boundary value problem)
even for 1D, existence or uniqueness is not guaranteed.
[see eigenvalue problem].
Nor is existence/uniqueness for 2D.

- Domains



$u_{xx} + u_{yy} = F$

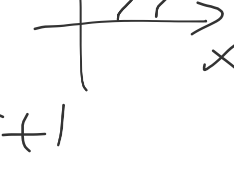
$(x, y) \longleftrightarrow (r, \theta)$, $u(x,y) = w(r, \theta)$

$\Delta u = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} = F$

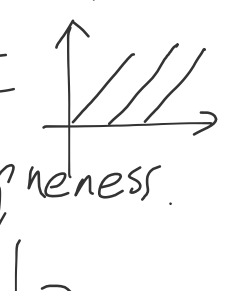
RK: Exercise.

Review. ch2

- characteristic curves $L(t)$
$\iff$ solutions
$x(t,s)$, $y(t,s)$
$u(t,s)$



- characteristics $L_p(t)$
[projection of characteristic curves] on x-y plane.



$u_x + u_y = u^2 + 1$

characteristics: [relations x-y] $\frac{dy}{dx} = \frac{1}{1}$

$y = x + c$

- Existence & uniqueness.

$J\big|_{\Gamma(s)} = \begin{vmatrix} a & b \\ (x_0)_s & (y_0)_s \end{vmatrix} = \cdots$

if $J \neq 0 \rightarrow$ unique solution

If $J = 0 \rightarrow \begin{cases} \text{no solution} \\ \text{or } \infty \text{ many solution.} \end{cases}$

- check one ✓ $\rightarrow \infty$ many
- contradiction $\rightarrow$ no solution