Lecture2 §1.4 – §1.5

Friday, April 3, 2020 9:25 AM

Objective { PDE as math models; PDE system (PDE + conditions) }

(I) Review.

- $\vec{F} = (F_1, F_2, F_3)$ in 3-D

$$\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

- Guass's thm.

$$\int_D \nabla\cdot\vec{F}\, dv = \int_{\partial D} \vec{F}\cdot\vec{n}\, ds$$

- $u_t - \Delta u = 0$ $\quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

2nd-order linear PDE.

$$u_t - (u_{xx} + u_{yy} + u_{zz}) = 0$$

(II) math models

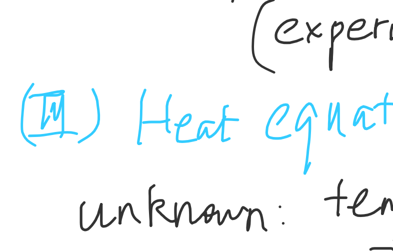
fundamental laws of physics. { laws of gravity; electromagnetism.; quantum mechanics }

math models { conservation of mass; — momentum; — of energy.; constitutive laws (experiments, assumption) }

(III) Heat equation (J. Fourier)

unknown: temperature $u(\vec{x}, t)$

$\vec{x} = (x, y, z)$



fixed domain D

$x \in D$

Derivation:

energy ⟷ internal energy. ⟷ temperature

conservation of energy

change in energy $t \to t+\Delta t$

$$\int_D [u(\vec{x}, t+\Delta t) - u(\vec{x}, t)]\, dV$$

$$= \int_t^{t+\Delta t}\int_D q(\vec{x}, t)\, dV\, dt$$

$$- \int_t^{t+\Delta t}\int_{\partial D} \vec{B}\cdot\vec{n}\, ds\, dt$$

q: rate of heat production.

$\vec{B}$: heat flux through ∂D

$\vec{n}$: unit outward normal

- $\div \Delta t$ take $\Delta t \to 0$

$$LHS = \lim_{\Delta t\to 0}\int_D \frac{u(\vec{x}, t+\Delta t) - u(\vec{x}, t)}{\Delta t}\, dV$$

$$= \int_D u_t\, dV$$

$$RHS = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\int_D q\, dV\, dt - \frac{1}{\Delta t}\int_t^{t+\Delta t}\int_{\partial D}\vec{B}\cdot\vec{n}\, ds\, dt$$

$$= \int_D q\, dV - \int_{\partial D}\vec{B}\cdot\vec{n}\, dS$$

$$\Rightarrow \int_D u_t\, dV = \int_D q\, dV - \int_{\partial D}\vec{B}\cdot\vec{n}\, ds$$

Guass ✓

$$= \int_D q\, dV - \int_D \nabla\cdot\vec{B}\, dV$$

$$= \int_D (q - \nabla\cdot\vec{B})\, dV$$

for any D.

$$\Rightarrow u_t = q - \nabla\cdot\vec{B}. \quad (1)$$

- constitutive law [Fourier's law]

$$\vec{B} = -k(\vec{x})\nabla u \quad (2)$$

$k > 0$: heat conduction coefficient.

- $(2) \to (1) \Rightarrow$

$$\boxed{u_t = q + \nabla\cdot(k\nabla u)}$$

heat equation.

RK: special case: k is constant, $q = 0$.

$$u_t = k\,\nabla\cdot(\nabla u)$$

$$= k\Delta u$$

$$= k(u_{xx} + u_{yy} + u_{zz})$$

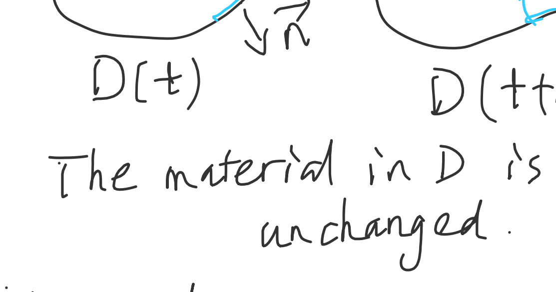
(IV) Hydrodynamics (fluid flow)

unknows

density $\rho(\vec{x}, t)$

velocity: $\vec{u}(\vec{x}, t)$

pressure: $P(\vec{x}, t)$



D(t) D(t+Δt)

The material in D is unchanged.

(i)

$$\frac{d}{dt}\int_D \rho\, dV = 0$$

$$\Rightarrow \int_D \rho_t\, dV + \int_{\partial D}\rho\,\vec{u}\cdot\vec{n}\, ds = 0$$

$$\int_D \rho_t\, dV + \int_D \nabla\cdot(\rho\vec{u})\, dV = 0$$

$$\int_D [\rho_t + \nabla\cdot(\rho\vec{u})]\, dV = 0$$

for all D.

$$\Rightarrow \rho_t + \nabla\cdot(\rho\vec{u}) = 0. \quad (1)$$

(ii) momentum.

$$\frac{d}{dt}\int_D \rho\vec{u}\, dV = -\int_{\partial D} P\vec{n}\, ds + \int_D \rho\vec{g}\, dV \quad (*)$$

forces { pressure. P; gravity (body force) }.

[Exercise, derive]

$$\Rightarrow \rho\vec{u}_t + \rho(\vec{u}\cdot\nabla)\vec{u} = -\nabla P + \rho\vec{g}, \quad (2)$$

(iii) constitutive laws

$$P = f(\rho) \quad \left(\begin{matrix}\text{property}\\ \text{of fluid}\end{matrix}\right) \quad (3)$$

Summary. (compressible fluid flow)

$$\begin{cases}\rho_t + \nabla\cdot(\rho\vec{u}) = 0 & (1)\\ \rho[\vec{u}_t + \vec{u}\cdot\nabla\vec{u}] = -\nabla P + \rho\vec{g} & (2)\\ P = f(\rho) & (3)\end{cases}$$

RK1 friction $\to + \mu\Delta\vec{u}$

RK2: incompressible fluid flow.

Navier-stokes

$$\begin{cases}\rho(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}) = \mu\Delta\vec{u} - \nabla P & (4)\\ \nabla\cdot\vec{u} = 0 & (5)\end{cases}$$

ρ = constant, (1) reduces to (5).

- other PDE

① wave Eq

$$u_{tt} = c^2\Delta u$$

② random motion.

$$\Delta u = -\frac{1}{k}$$

③ minimal surface

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0$$

(V) Associated conditions.

PDE system < PDE; conditions

(IC) initial conditions. $(t = t_0)$

(BC) Boundary conditions (∂D)

$$BC\begin{cases}u(x, y, z, t) = f_1(x, y, z, t) & \text{on } \partial D\\ \partial_n u = f_2, & \text{on } \partial D\\ \text{or: } a\cdot\partial_n u + u = h & \text{on } \partial D\end{cases}$$

$$IC: \; u(x, y, z, t_0) = u_0(x, y, z)$$

To do list

HW 1.4a

Next: chapter 2.