Lecture3 §2.1 – §2.3

Wednesday, April 8, 2020 9:24 AM

Objective { First-order PDE; method of characteristics

(I) Review

- Euler model for fluid flow.

mass conservation

$\rho_t + \nabla\cdot(\rho\vec{u}) = 0$

- general: consider concentration C

$C_t + \nabla\cdot(C\vec{u}) = 0$

convection Equation (Eq)

for $C(\vec{x}, t)$

1st-order Eq.

(II) 1st order PDE.

- General form.

$F(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots u_{x_n}) = 0$

$u(x_1, x_2, \cdots, x_n)$ unknown.

- we focus on $n=2$

$u = u(x,y)$

domain $(x,y) \in D \subset \mathbb{R}^2$

$F(x, y, u, u_x, u_y) = 0$

The graph of u is a surface in $\mathbb{R}^3$

(III) Quasilinear Equation

- $a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$ (*)

- semi-linear

$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y,u)$

- linear:

$a(x,y)\,u_x + b(x,y)\,u_y = c_1(x,y) + c_0(x,y)\cdot u$

- Example:

$u_x = c_0 u + c_1(x,y)$

$a=1,\ b=0,\ c_0$ is constant.

[RK: review ODE.]

$I = e^{c_0 x}$

$u(x,y) = e^{c_0 x}\left[\int_0^x e^{c_0\xi} c_1(\xi, y)\,d\xi + C(y)\right]$ (**)

Initial condition

Case 1: $u(0,y) = y$

By (**): $u(0,y) = e^0[0 + C(y)] = y$

$\Rightarrow C(y) = y$

So $u(x,y) = e^{c_0 x}\left[\int_0^x e^{c_0\xi} c_1(\xi, y)\,d\xi + y\right]$

unique solution.

case 2: $u(x,0) = x$,

we consider $c_1 = 0$.

$u(x,0) = e^{c_0 x}[0 + C(0)]$

$= C(0)\cdot e^{c_0 x}$

$\Rightarrow C(0)\,e^{c_0 x} = x$ for all x

Contradiction.

No solution!

RK: if $u(x,0) = e^{c_0 x}$ and $c_1 = 0$.

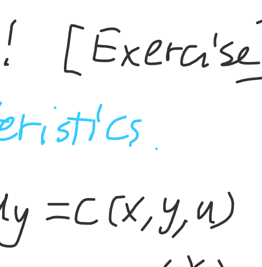
∞ many solutions! [Exercise]

(IV) method of characteristics

$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$ (*)

Derivation

parameter t.



$x = x(t),\ y = y(t)$

$u = u(x(t), y(t))$

set $x_t = a,\ y_t = b$

chain rule:

$u_t = u_x x_t + u_y y_t$

$= a u_x + b u_y$

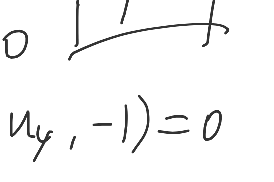
$\overset{(*)}{=} c$

$\Rightarrow$ characteristic equations (CE)

CE: $\begin{cases} x_t = a(x,y,u) \\ y_t = b(x,y,u) \\ u_t = c(x,y,u) \end{cases}$

RK1: It is autonomous system.

RK2: geometric intepretation



$a u_x + b u_y = c$

$\Rightarrow a u_x + b u_y - c = 0$

$(a, b, c)\cdot(u_x, u_y, -1) = 0$

$\vec{n} = (u_x, u_y, -1)$ is normal vector

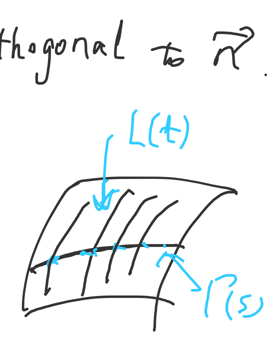
$\vec{T} = (x_t, y_t, u_t) \overset{CE}{=} (a, b, c)$

$\Rightarrow \vec{T} \perp \vec{n}$ $\vec{T}$ is orthogonal to $\vec{n}$.

Initial condition.

$u(x,0) = f(x)$

or $u(0,y) = g(y)$



or More general.

$x = x(t,s),\ y = y(t,s),\ u = u(t,s)$

at $t=0$. $\Gamma(s)$

IC: $\begin{cases} x(0,s) = x_0(s) \\ y(0,s) = y_0(s) \\ u(0,s) = u_0(s). \end{cases}$

Summary:

✓ • system: CE + IC

✓ • solution: $(x(t,s), y(t,s), u(t,s))$

- convert $u = u(x,y)$.

$(t,s) \longleftrightarrow (x,y)$?

convertible if

$J = \begin{vmatrix} x_t & y_t \\ x_s & y_s \end{vmatrix} \neq 0$

unique solution.

(V) Example

① $u_x + u_y = 2$

$u(x,0) = e^x \rightarrow u(s,0) = e^s$

solution: step 1:

CE $\begin{cases} x_t = a = 1 & (1) \\ y_t = b = 1 & (2) \\ u_t = c = 2 & (3). \end{cases}$

IC: $\begin{cases} x(0,s) = s & \text{(assume) } (4) \\ y(0,s) = 0 & (5) \end{cases}$

at $t=0$ $u(0,s) = u\big|_{x = x(0,s),\ y = y(0,s)}$

$= u\big|_{x=s,\ y=0}$

$= e^s$ (6)

step 2: (1) $\Rightarrow x(t,s) = \int_0^t 1\,dt + f_1(s)$

$= t + f_1(s)$

use (4) $\Rightarrow x(0,s) = f_1(s) \overset{(4)}{=} s$

$x(t,s) = t + s$

(2) (5) $\Rightarrow y = \int_0^t 1\,dt + y(0,s)$

$= t + 0$

$= t$

(3) (6) $\Rightarrow u(t,s) = \int_0^t 2\,dt + u(0,s)$

$= 2t + e^s$

So $x = t+s,\ y = t,\ u = 2t + e^s$

step 3: convert $(s,t) \longleftrightarrow (x,y)$.

$t = y,\ s = x - t = x - y$

$\Rightarrow u = 2y + e^{x-y}$

unique solution

To do list:

① HW#3

② Exercise.