Lecture18

Wednesday, June 3, 2020 9:24 AM

§9.4.2

Objective: wave Eq in $\mathbb{R}^3$ (general case)

(I) Wave Eq in $\mathbb{R}^3$

$$(*)\quad \begin{cases} u_{tt} = c^2 \Delta u, & \vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3,\ t \geq 0 \\ u(\vec{x}, 0) = f(\vec{x}), & \vec{x} \in \mathbb{R}^3 \\ u_t(\vec{x}, 0) = g(\vec{x}), & \vec{x} \in \mathbb{R}^3. \end{cases}$$

Claim: It is enough to solve a problem with $f = 0$

Thm: If

$$(*)_1\quad \begin{cases} u_{tt} = c^2 \Delta u & \text{(PDE)} \\ u(\vec{x}, 0) = 0 & \text{(IC)} \\ u_t(\vec{x}, 0) = g(\vec{x}) \end{cases}$$

then $v = u_t$ solves

$$\begin{cases} v_{tt} = c^2 \Delta v \\ v(\vec{x}, 0) = g(\vec{x}) \\ v_t(\vec{x}, 0) = 0 \end{cases}$$

Proof: $v = u_t$.

take $\frac{\partial}{\partial t}$ on PDE of u.

$$u_{ttt} = c^2 \frac{\partial}{\partial t}(\Delta u)$$

$$(u_t)_{tt} = c^2 \Delta (u_t)$$

$$\Rightarrow v_{tt} = c^2 \Delta v$$

$$v(\vec{x}, 0) = u_t(\vec{x}, 0) = g(\vec{x})$$

$$v_t(\vec{x}, 0) = u_{tt}(\vec{x}, 0) \overset{\text{(PDE)}}{=} c^2 \Delta u(\vec{x}, 0) = c^2 \Delta(0) = 0$$

RK: If $U_g(\vec{x}, t)$ is solution of $(*)_1$, then $\frac{\partial}{\partial t} U_f$ is a solution of

$$(*)_2\quad \begin{cases} u_{tt} = c^2 \Delta u \\ u(\vec{x}, 0) = f \\ u_t(\vec{x}, 0) = 0 \end{cases}$$

For $(*)$, we get

$$\boxed{u(\vec{x}, t) = U_g(\vec{x}, t) + \frac{\partial}{\partial t}(U_f)}$$

RK2: D'Alembert's formula in 1-D can be put in above form □

(II) Darboux Eq.

Given a smooth function $h(\vec{x})$ in $\mathbb{R}^3$ [e.g. $h(\vec{x}) = u(\vec{x}, t)$]

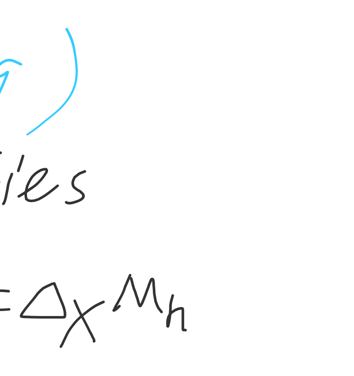
The spherical mean of radius a around $\vec{x}$

$$M_h(a, \vec{x}) = \frac{1}{4\pi a^2} \int_{|\vec{\xi} - \vec{x}| = a} h(\vec{\xi})\, ds_\xi$$

$\vec{\xi} - \vec{x} = a\vec{\eta}$

$$= \frac{1}{4\pi} \int_{|\vec{\eta}| = 1} h(\vec{x} + a\vec{\eta})\, ds_\eta$$

$|\vec{\xi} - \vec{x}| = a$



By continuity

$$\lim_{a \to 0} M_h(a, \vec{x}) = h(\vec{x})$$

$$\lim_{a \to 0} M_u(a, \vec{x}) = u(\vec{x})$$

Thm: (Darboux Eq)

$M_h(a, \vec{x})$ satisfies

$$\left(\frac{\partial^2}{\partial a^2} + \frac{2}{a}\frac{\partial}{\partial a}\right) M_h = \Delta_x M_h$$

Proof: By def & Guass's formula

$$\frac{\partial}{\partial a} M_h = \frac{1}{a^2} \Delta_x \int_0^a M_h(r, \vec{x}) r^2\, dr$$

[Exercise, see the solution of Ex 9.7]

$$\frac{\partial^2}{\partial a^2} M_h = \frac{\partial}{\partial a}\left[\frac{1}{a^2} \Delta_x \int_0^a M_h(r, \vec{x}) r^2\, dr\right]$$

$$= \frac{-2}{a^3} \Delta_x \int_0^a M_h(r, \vec{x}) r^2\, dr + \frac{1}{a^2} \Delta_x \left(M_h(a, \vec{x}) a^2\right)$$

$$= \frac{-2}{a^3} \cdot a^2 \frac{\partial}{\partial a} M_h + \Delta_x (M_h)$$

$$= \frac{-2}{a} \frac{\partial}{\partial a} M_h + \Delta_x (M_h)$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial a^2} M_h + \frac{2}{a}\frac{\partial}{\partial a} M_h = \Delta_x (M_h)}$$

(III) General solution of M_u

Thm: If u satisfies

$$(*)_1\quad \begin{cases} u_{tt} = \Delta u, & \vec{x} \in \mathbb{R}^3,\ t > 0. \\ u(\vec{x}, 0) = 0 & \vec{x} \in \mathbb{R}^3 \\ u_t(\vec{x}, 0) = g(\vec{x}) \end{cases}$$

then M_u satisfies

$$\begin{cases} \frac{\partial^2}{\partial t^2} M_u(a, \vec{x}, t) = \left(\frac{\partial^2}{\partial a^2} + \frac{2}{a}\frac{\partial}{\partial a}\right) M_u & (\text{PDE})_{M_u} \\ M_u(a, \vec{x}, 0) = 0 \\ \frac{\partial}{\partial t} M_u(a, \vec{x}, 0) = M_g(a, \vec{x}) \end{cases}$$

Proof: By Darboux Eq.

$$\left(\frac{\partial^2}{\partial a^2} + \frac{2}{a}\frac{\partial}{\partial a}\right) M_u = \Delta_x M_u$$

$$\overset{\text{def}}{=} \Delta_x \left[\frac{1}{4\pi} \int_{|\eta| = 1} u(\vec{x} + a\vec{\eta}, t)\, ds_\eta\right]$$

$$= \frac{1}{4\pi} \int_{|\eta| = 1} \Delta_x u(\vec{x} + a\vec{\eta}, t)\, ds_\eta$$

$$\overset{(*)_1}{=} \frac{1}{4\pi} \int_{|\eta| = 1} \frac{\partial^2}{\partial t^2} u(\vec{x} + a\vec{\eta}, t)\, ds_\eta$$

$$= \frac{\partial^2}{\partial t^2} \left[\frac{1}{4\pi} \int_{|\eta| = 1} u(\vec{x} + a\vec{\eta}, t)\, ds_\eta\right]$$

$$= \frac{\partial^2}{\partial t^2} M_u \Rightarrow (\text{PDE})_{M_u}$$

$(IC)_1$: $M_u(a, \vec{x}, 0)$

$$= \frac{1}{4\pi} \int_{|\eta| = 1} u(\vec{x} + a\vec{\eta}, 0)\, ds_\eta \overset{(IC)_u}{=} 0$$

$(IC)_2$: $\frac{\partial}{\partial t} M_u(a, \vec{x}, 0)$

$$= \frac{\partial}{\partial t}\left[\frac{1}{4\pi} \int_{|\eta| = 1} u(\vec{x} + a\vec{\eta}, t)\, ds_\eta\right]\Bigg|_{t=0}$$

$$= \frac{1}{4\pi} \int_{|\eta| = 1} \frac{\partial u}{\partial t}(\vec{x} + a\vec{\eta}, t)\Big|_{t=0} ds_\eta$$

$$\overset{(*)_1}{=} \frac{1}{4\pi} \int_{|\eta| = 1} g(\vec{x} + a\vec{\eta})\, ds_\eta$$

$$= M_g(a, \vec{x}). \quad \square$$

treat $\vec{x}$ as parameter

$M_u(a, t)$, $a \to r$

$M_u(a, \vec{x}, t)$

$$= \frac{1}{2ac} \int_{a-ct}^{a+ct} s \tilde{M}_g(s, \vec{x})\, ds \qquad (1)$$

where $\tilde{M}_g$ is even extension of M_g

RK: $\int_{a-ct}^{ct-a} s \tilde{M}_g(s)\, ds = 0$

since $s \tilde{M}_g(s)$ is odd function

$$\int_{a-ct}^{a+ct} = \int_{a-ct}^{ct-a} + \int_{ct-a}^{a+ct} = \int_{a-ct}^{ct-a} + \int_{ct-a}^{ct+a}$$

(1) $\Leftrightarrow$

$$M_u = \frac{1}{2ac} \int_{ct-a}^{ct+a} s \tilde{M}_g(s, \vec{x})\, ds \qquad (2)$$

(IV) General solution of u.

$$u(\vec{x}, t) = \lim_{a \to 0} M_u(a, \vec{x}, t)$$

$$= \frac{1}{2ac}\, ct\, \tilde{M}_g(ct, \vec{x}) \cdot 2a$$

$$= t\, \tilde{M}_g(ct, \vec{x}).$$

$\boxed{ct > 0}$ $\Rightarrow$ $= t\, M_g(ct, \vec{x})$

RK: This is $U_g(\vec{x}, t)$

$\Rightarrow U_f = t\, M_f(ct, \vec{x})$

Summary:

For $(*)$ with f & g

$$u(\vec{x}, t) = t\, M_g(ct, \vec{x}) + \frac{\partial}{\partial t}\left[t\, M_f(ct, \vec{x})\right]$$

$$= \frac{1}{4\pi c^2 t} \int_{|\vec{\xi} - \vec{x}| = ct} g(\vec{\xi})\, ds_\xi + \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \int_{|\vec{\xi} - \vec{x}| = ct} f(\vec{\xi})\, ds_\xi\right]$$

physics: $f = 0$