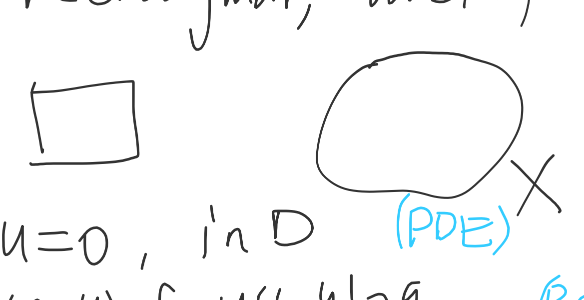
Lecture12
Wednesday, May 13, 2020 9:22 AM

§7.7.1

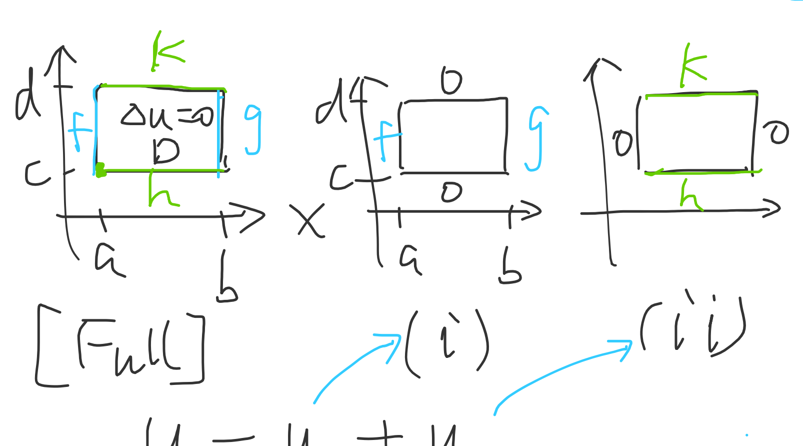
Objective: separation of variables for $\Delta u = 0$.

(I) Dirichlet problem.

for simple domains (rectangular, disk)



$$\begin{cases} \Delta u = 0, \text{ in } D & \text{(PDE)} \\ u(a,y)=f,\ u(b,y)=g & (BC)_1 \\ u(x,c)=h,\ u(x,d)=k & (BC)_2 \end{cases}$$



[Full] $\to$ (i), (ii)

$$u = u_1 + u_2$$

(i) $\begin{cases} \Delta u_1 = 0 \\ u_1(a,y)=f,\ u_1(b,y)=g \\ u_1(x,c)=u_1(x,d)=0 \end{cases}$

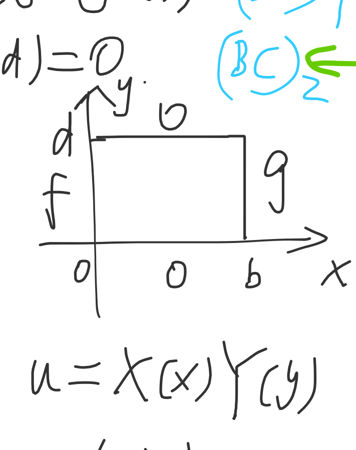
(ii) $\begin{cases} \Delta u_2 = 0 & \text{(PDE)} \\ u_2(a,y)=u_2(b,y)=0, & (BC)_1 \\ u_2(x,c)=h,\ u_2(x,d)=k, & (BC)_2 \end{cases}$

(i) & (ii) can be solved by separation of variable.

$u_1 = \sum X_n Y_n$, $u_2 = \sum X_n Y_n$

- Solve (i) with $a=c=0$.

$$\begin{cases} \Delta u = 0 & \text{(PDE)} \\ u(0,y)=f(y),\ u(b,y)=g(y), & (BC)_1 \\ u(x,0)=0,\ u(x,d)=0 & (BC)_2 \end{cases}$$



step 1: Try solution $u = X(x)Y(y)$

PDE $\Rightarrow (XY)_{xx} + (XY)_{yy} = 0$

$$X_{xx}Y + XY_{yy} = 0$$

$$-\frac{X_{xx}}{X} = \frac{Y_{yy}}{Y} = -\lambda$$

$$\begin{cases} X_{xx} = \lambda X & (1) \\ Y_{yy} = -\lambda Y & (2) \end{cases}$$

$(BC)_2 \Rightarrow \begin{cases} X(x)Y(0)=0 \\ X(x)Y(d)=0 \end{cases}$

$\Rightarrow Y(0) = Y(d) = 0.$ (3)

$$\begin{cases} Y_{yy} = -\lambda Y & (2) \\ Y(0)=Y(d)=0 & (3) \end{cases}$$

Eigenvalue problem

$$Y_n = \sin\left(\frac{n\pi}{d}y\right),\quad n=1,2,3,\dots$$
$$\lambda_n = \left(\frac{n\pi}{d}\right)^2,$$

RK: compare with wave Eq.

$u_{xx}+u_{yy}=0$	$u_{tt}-u_{xx}=0$
$u(x,0)=u(x,d)=0$	$u(0,t)=u(L,t)=0$
$x \to t,\ y \to x$	$X_n = \sin\left(\frac{n\pi}{L}x\right)$
$d \to L$, $Y_n \to X_n$	

with $\lambda_n = \left(\frac{n\pi}{d}\right)^2$, in (1),

$$X'' = \left(\frac{n\pi}{d}\right)^2 X$$

$$X = X_n = a_n e^{\frac{n\pi}{d}x} + b_n e^{-\frac{n\pi}{d}x}$$

we rewrite it as

$$X_n = A_n \sinh\left(\frac{n\pi}{d}x\right) + B_n \sinh\left[\frac{n\pi}{d}(x-b)\right]$$

RK: $\sinh(z) = \frac{e^z - e^{-z}}{2}$

$\cosh(z) = \frac{e^z + e^{-z}}{2}$

Step 2: $u = \sum X_n Y_n$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{d}y\right)\cdot \left[A_n \sinh\left(\frac{n\pi x}{d}\right) + B_n \sinh\left[\frac{n\pi}{d}(x-b)\right]\right]$$

step 3: Find A_n & B_n.

By $(BC)_1$

$$f(y) = u(0,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{d}y\right)\cdot B_n \sinh\left(-\frac{n\pi}{d}b\right)$$

$$B_n \sinh\left(-\frac{n\pi}{d}b\right) = \frac{2}{d}\int_0^d f(y)\sin\left(\frac{n\pi}{d}y\right)dy$$

(assign) $\triangleq \alpha_n$ [Fourier coefficients of $f(y)$]

$$B_n = \frac{\alpha_n}{\sinh\left(-\frac{n\pi}{d}b\right)} = -\frac{\alpha_n}{\sinh\left(\frac{n\pi}{d}b\right)}.$$

$$g(y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi b}{d}\right)\sin\left(\frac{n\pi y}{d}\right)$$

$$A_n \sinh\left(\frac{n\pi b}{d}\right) = \frac{2}{d}\int_0^d g(y)\sin\left(\frac{n\pi y}{d}\right)dy = \beta_n$$

$$\Rightarrow A_n = \frac{\beta_n}{\sinh\left(\frac{n\pi b}{d}\right)}$$

HW 11: Solve (ii) with $a=c=0$.

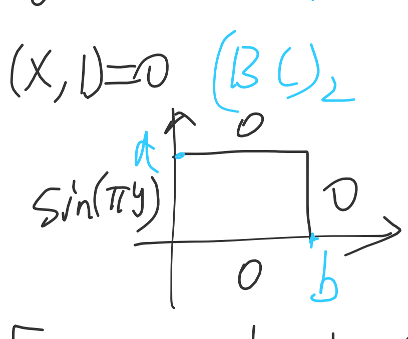
interchange $x \leftrightarrow y$
$X \leftrightarrow Y$

(II) Examples:

① $\begin{cases} \Delta u = 0 \\ u(0,y) = \sin(\pi y), & (BC)_1 \\ u(1,y) = 0 \\ u(x,0) = u(x,1) = 0 & (BC)_2 \end{cases}$

$b=1$, $d=1$

$D = [0,b] \times [0,d]$



- $d=1$. [L=1, chapter 5]

$Y_n = \sin(n\pi y)$

$$u = \sum_{n=1}^{\infty} \sin(n\pi y)\cdot\left[A_n \sinh(n\pi x) + B_n \sinh[n\pi(x-1)]\right]$$

($b=1$)

$$u(0,y) = \sum_{n=1}^{\infty} \sin(n\pi y)\, B_n \sinh(-n\pi) = \sin(\pi y)$$

$$B_n \sinh(-n\pi) = \begin{cases} 1, & n=1 \\ 0, & n\neq 1 \end{cases}$$

$$B_1 = \frac{1}{\sinh(-\pi)} = -\frac{1}{\sinh(\pi)}$$

$B_n = 0,\ n \neq 1$

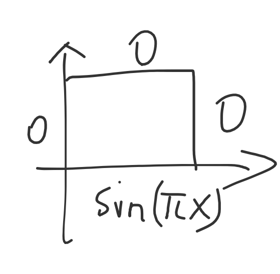
$$u(1,y) = \sum_{n=1}^{\infty} \sin(n\pi y)\, A_n \sinh(n\pi) = 0$$

$\Rightarrow A_n = 0,\quad n=1,2,\dots$

$$u = -\frac{1}{\sinh(\pi)}\sin(\pi y)\cdot\sinh(\pi(x-1))$$

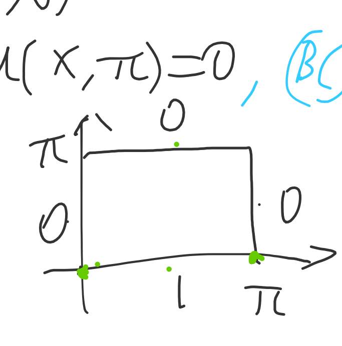
② [Example 7.23]

$$\begin{cases} \Delta u = 0 \\ u(0,y)=u(1,y)=0 \\ u(x,0)=\sin(\pi x) \\ u(x,1)=0 \end{cases}$$



[Exercise]

③ $\begin{cases} \Delta u = 0 \\ u(0,y)=u(\pi,y)=0 & \leftarrow (BC)_1 \\ u(x,0)=1,\ u(x,\pi)=0, & (BC)_2 \end{cases}$



- $(BC)_1$ $b=\pi$ [$L=\pi$, ch 5]

$$X_n = \sin\left(\frac{n\pi}{b}x\right) = \sin(nx)$$
$n=1,2,3$

$$u = \sum_{n=1}^{\infty}\sin(nx)\left[A_n\sinh(ny) + B_n\sinh[n(y-\pi)]\right]$$

$(BC)_2 \Rightarrow$

$$0 = u(x,\pi) = \sum \sin(nx)\, A_n \sinh(n\pi)$$

$A_n = 0,\quad n=1,2,\dots$

$u(x,0)=1 \Rightarrow$

$$1 = \sum_{n=1}^{\infty}\sin(nx)\cdot B_n\sinh(-n\pi)$$

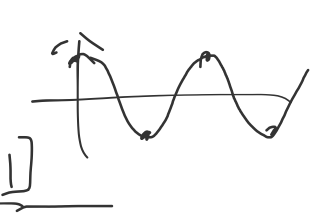
$$B_n\cdot\sinh(-n\pi) = \frac{2}{\pi}\int_0^{\pi} 1\cdot\sin(nx)\,dx$$

[Fourier coefficients]

$$= -\frac{2}{\pi}\frac{1}{n}\cos(nx)\Big|_0^{\pi}$$

$$= \frac{-2}{\pi n}\left[\cos(n\pi)-1\right]$$

$$= \frac{-2}{\pi n}\left[(-1)^n - 1\right]$$



$$\Rightarrow B_n = \frac{-2[(-1)^n-1]}{\sinh(-n\pi)\cdot n\pi}$$

$\sinh(-z) = -\sinh(z)$

$$= \frac{2[(-1)^n-1]}{n\pi\sinh(n\pi)}.$$

$$u = \sum_{n=1}^{\infty}\frac{2[(-1)^n-1]}{n\pi\sinh(n\pi)}\sin(nx)\cdot\sinh[n(y-\pi)]$$

RK: If n is even, $B_n = 0$

u can be further simplified, [Exercise, textbook]

To do list

HW 11

Next §7.7.2