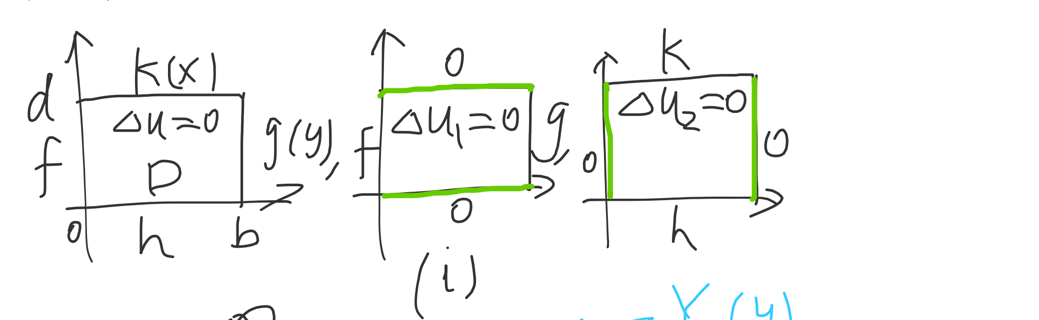
§7.7.2

Objective: separation of variables (circular domain)

(I) Review



(i) $u=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{d}y\right)\cdot$ ← $Y_n(y)$

$\left[A_n\sinh\left(\frac{n\pi}{d}x\right)+B_n\sinh\left(\frac{n\pi}{d}(x-b)\right)\right]$

$A_n\ B_n \leftarrow f, g$

(ii) $u=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{b}x\right)\cdot$ ← $X_n(x)$

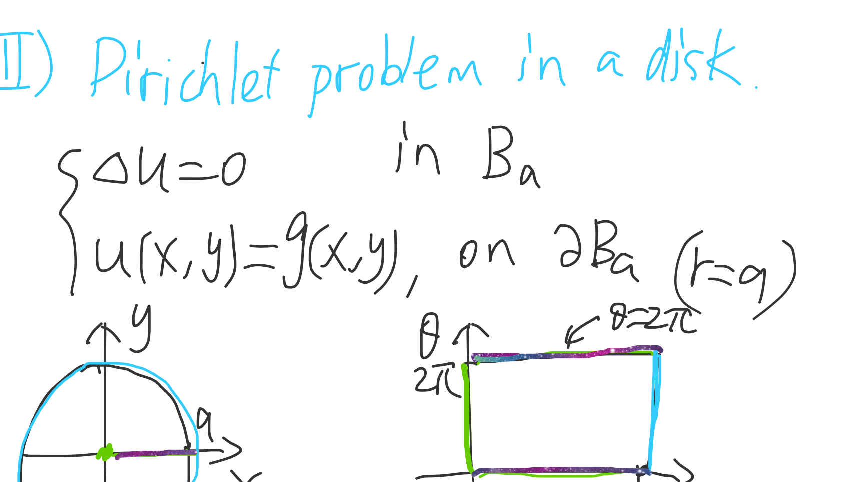
$\left[A_n\sinh\left(\frac{n\pi}{b}y\right)+B_n\sinh\left(\frac{n\pi}{b}(y-d)\right)\right]$

$A_n, B_n \leftarrow h, k.$

[in HW, show derivations].

(II) Dirichlet problem in a disk.

$$\begin{cases}\Delta u=0 & \text{in } B_a\\ u(x,y)=g(x,y), & \text{on } \partial B_a\ (r=a)\end{cases}$$



$B_a: 0\le r\le a$
$0\le\theta\le 2\pi.$

$w(r,\theta)=u(x(r,\theta),y(r,\theta))$

$$\begin{cases}\Delta w=w_{rr}+\frac{1}{r}w_r+\frac{1}{r^2}w_{\theta\theta}=0, \text{ in } B_a & \text{(PDE)}\\ w(a,\theta)=g(x(a,\theta),y(a,\theta))\\ \quad =h(\theta)., \quad \text{at } r=a & (BC)_1\\ \lim_{r\to0}w(r,\theta) \text{ exists \& finite}, & (BC)_2\\ w(r,0)=w(r,2\pi), \\ w_\theta(r,0)=w_\theta(r,2\pi) & (BC)_3\\ \text{[Periodic BC]}\end{cases}$$

RK: $r=0$, $\theta=0, 2\pi$, are created boundaries due to transform $(x,y)\longleftrightarrow(r,\theta)$

Solution:
step(1a) $w=R(r)\Theta(\theta)$

PDE $\Rightarrow R_{rr}\Theta+\frac{1}{r}R_r\Theta+\frac{1}{r^2}R\Theta_{\theta\theta}=0.$

$r^2R_{rr}\Theta+rR_r\Theta=-R\Theta_{\theta\theta}$

$\frac{r^2R_{rr}}{R}+\frac{rR_r}{R}=-\frac{\Theta_{\theta\theta}}{\Theta}=\lambda$

$$\begin{cases}r^2R_{rr}+rR_r=\lambda R, & (1)\\ \Theta_{\theta\theta}=-\lambda\Theta, & (2)\end{cases}$$

$(BC)_3\Rightarrow\begin{cases}R(r)\Theta(0)=R(r)\Theta(2\pi)\\ R(r)\Theta_\theta(0)=R(r)\Theta_\theta(2\pi)\end{cases}$

$R(r)\ne0$
$\Rightarrow \Theta(0)=\Theta(2\pi),\ \Theta_\theta(0)=\Theta_\theta(2\pi),\ (3)$

(2)(3) Eigenvalue problem for $\Theta(\theta)$ with periodic BC.

$$\begin{cases}\Theta_n(\theta)=A_n\cos(n\theta)+B_n\sin(n\theta).\\ \lambda_n=n^2, \quad n=0,1,2,\cdots\end{cases}$$

step(1b) substitute $\lambda_n=n^2$ into (1)

$r^2R_{rr}+rR_r=n^2R$
[review math146b]

$$R=R_n=\begin{cases}C_0+D_0\ln(r) & ,n=0\\ C_nr^n+D_nr^{-n}, & n=1,2,\cdots\end{cases}$$

$(BC)_2\Rightarrow \lim_{r\to0}w$ is finite.

$\Rightarrow D_0=0,\ D_n=0,\ n=1,2,\cdots$

So $R_n(r)=\begin{cases}C_0, & n=0\\ C_nr^n, & n=1,2,\cdots\end{cases}$

Step2:
$w=\sum_{n=0}^{\infty}R_n\Theta_n$

$=A_0C_0+\sum_{n=1}^{\infty}C_nr^n.$
$[A_n\cos(n\theta)+B_n\sin(n\theta)]$

$=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}r^n[\alpha_n\cos(n\theta)+\beta_n\sin(n\theta)]$ ←

RK: $\frac{\alpha_0}{2}=A_0C_0$, $\alpha_n=A_n\cdot C_n$
$\beta_n=B_n\cdot C_n$

Step3:
$(BC)_1\Rightarrow w(a,\theta)=h(\theta).$

$w(a,\theta)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}a^n[\alpha_n\cos(n\theta)+\beta_n\sin(n\theta)]$
$=h(\theta).$

$\alpha_n=\frac{1}{a^n}\frac{1}{\pi}\int_0^{2\pi}h(\theta)\cos(n\theta)\,d\theta$

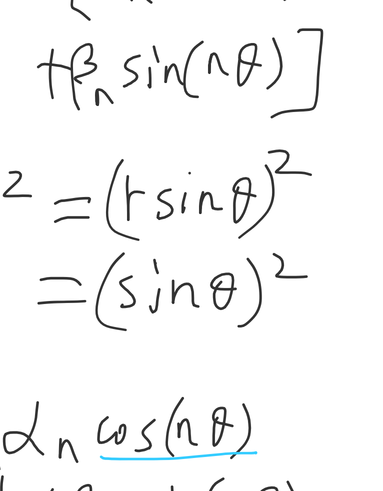
$\beta_n=\frac{1}{a^n}\frac{1}{\pi}\int_0^{2\pi}h(\theta)\sin(n\theta)\,d\theta$

[Fourier coefficients]

(III) Example:

$$\begin{cases}\Delta w=0 & \text{in } 0<r<1\\ w(1,\theta)=y^2, & \text{on } r=1\end{cases}$$

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$$ ←



$w(r,\theta)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}r^n[\alpha_n\cos(n\theta)+\beta_n\sin(n\theta)]$

on $r=1$, $h(\theta)=y^2=(r\sin\theta)^2$
$=(\sin\theta)^2$

$w(1,\theta)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\alpha_n\cos(n\theta)+\beta_n\sin(n\theta).$
$=h(\theta)=\sin^2\theta$
$=\frac{1}{2}-\frac{1}{2}\cos(2\theta).$

$\Rightarrow\alpha_n=\begin{cases}1, & n=0\\ -\frac{1}{2}, & n=2\\ 0, & n\ne0,2\end{cases}$

$\beta_n=0,\ n=1,2,\cdots$

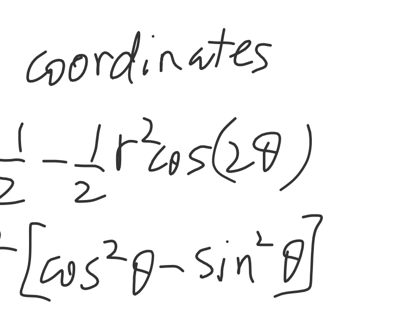
$w=\frac{1}{2}-\frac{1}{2}r^2\cos(2\theta)$ ←

RK1: In cartesian coordinates
$u(x,y)=w(r,\theta)=\frac{1}{2}-\frac{1}{2}r^2\cos(2\theta)$
$=\frac{1}{2}-\frac{1}{2}r^2[\cos^2\theta-\sin^2\theta]$
$=\frac{1}{2}-\frac{1}{2}(x^2-y^2)$

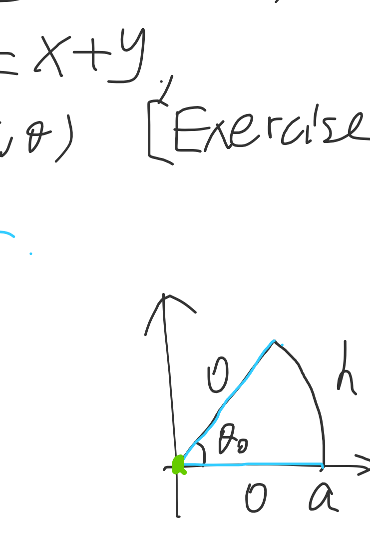
RK2: $w(1,\theta)=x+y$,
Find $w(r,\theta)$ [Exercise]

(III) A sector.

$D(a,\theta_0)$:
$0<r<a$
$0<\theta<\theta_0$



$$\begin{cases}\Delta w=0 & \text{in } D(a,\theta_0)\\ w(a,\theta)=h(\theta), \quad r=a & (BC)_1\\ \lim_{r\to0}w(r,\theta) \text{ is finite.} & (BC)_2\\ w(r,0)=w(r,\theta_0)=0. & (BC)_3\end{cases}$$



$(BC)_3\rightarrow\Theta_n(\theta)=\sin\left(\frac{n\pi}{\theta_0}\theta\right)$
$n=1,2,3,\cdots$

$w(r,\theta)=\sum_{n=1}^{\infty}\alpha_n\sin\left(\frac{n\pi}{\theta_0}\theta\right)\cdot r^{n\pi/\theta_0}$

RK: [Exercise] see textbook.

(IV) Poisson's formula.

in (II): substitute α_n, β_n into w

$w(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}h(\varphi)\,d\varphi$
$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\int_0^{2\pi}[\cos(n\varphi)\cos(n\theta)+\sin(n\varphi)\sin(n\theta)]h(\varphi)\,d\varphi$

$=\frac{1}{\pi}\int_0^{2\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos(n(\theta-\varphi))\right]h(\varphi)\,d\varphi$ ← BC.

$\Rightarrow w(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}K(r,\theta,a,\varphi)h(\varphi)\,d\varphi$

$K(r,\theta,a,\varphi)=\frac{a^2-r^2}{a^2-2ar\cos(\theta-\varphi)+r^2}$

Poisson's kernel.

→ Poisson's formula.

RK1: symmetry:
$K(r,\theta,a,\varphi)=\tilde{K}(r,a,\theta-\varphi)$

RK2: Let $r\to0\Rightarrow K=1$

$w(0,\theta)=\frac{1}{2\pi}\int_0^{2\pi}h(\varphi)\,d\varphi$
$=\frac{1}{2\pi}\int_0^{2\pi}w(a,\varphi)\,d\varphi.$

[Mean value Principle].

To do list.
HW12

Next: ch8