Lecture14

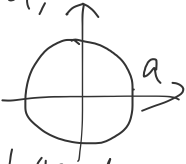
Wednesday, May 20, 2020 9:24 AM

§8.2

Objective
- review
- δ-function
- fundamental solution
- Green's representation formula.

(I) Review

$$\begin{cases} \Delta w = 0 & \text{in a disk } D. \\ w = h(\theta), & \text{on } r=a, \end{cases}$$

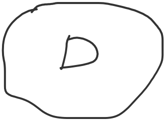


$$w(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} K(r,\theta,a,\varphi)\,h(\varphi)\,d\varphi$$

where $K = \dfrac{a^2-r^2}{a^2-2ar\cos(\theta-\varphi)+r^2}$

Q: can we derive integral solution for general case?

$$\begin{cases} \Delta u = f & \text{in } D \\ u = g & \text{on } \partial D \end{cases}$$



(II): fundamental solution.

recall: $\Delta u = 0$.

fundamental solution.

$$u = -\frac{1}{2\pi}\ln(r)$$

denote: it as

$$\Gamma(x,y) = -\frac{1}{2\pi}\ln(r)$$

$$= -\frac{1}{2\pi}\ln\left(\sqrt{x^2+y^2}\right)$$

$$= -\frac{1}{4\pi}\ln\left(x^2+y^2\right)$$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

verify if $\Delta\Gamma = 0$?

$$[\ln(r)]' = \frac{1}{r}, \quad [\ln r]'' = \frac{-1}{r^2}$$

$$\Delta[\ln(r)] = \underline{\frac{-1}{r^2}} + \frac{1}{r}\cdot\frac{1}{r} + 0$$

$$= 0 \qquad \text{if } r \neq 0.$$

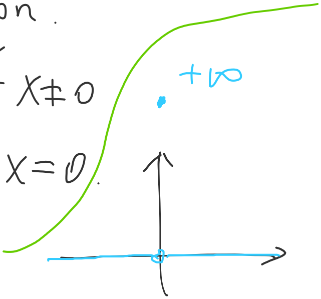
At $r=0$, there is singularity!

Is it 0 or ∞ ?

$$\Longleftrightarrow \begin{cases} \Delta\Gamma(x,y) = 0, & \text{if } (x,y) \neq (0,0) \\ \Delta\Gamma(x,y) \text{ is singular at } (0,0). \end{cases}$$

(III) δ-function.

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0. \end{cases}$$



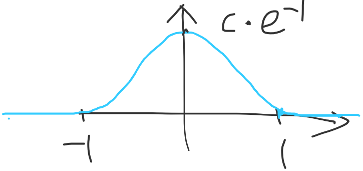
$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

It is not a classical function.

It is a distribution called Dirac distribution

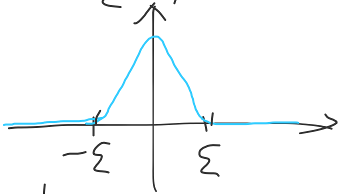
- approximate $\delta(x)$

$$\rho(x)=\begin{cases} c\cdot e^{\frac{1}{|x|^2-1}}, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$$



c is a constant chosen such that $\int_{-1}^{1}\rho(x)\,dx=1$

$$\rho_\varepsilon(x)=\frac{1}{\varepsilon^2}\rho\left(\frac{x}{\varepsilon}\right), \qquad \varepsilon>0$$



$$\int \rho_\varepsilon(x)\,dx=1$$

$$\rho_\varepsilon(x)\longrightarrow\delta(x), \text{ as } \varepsilon\to 0.$$

Note: $\int \rho_\varepsilon(x)\,dx = 1 \Rightarrow \int \delta(x)\,dx = 1$

- a shift:

$$\delta(x-\xi) = \begin{cases} \infty, & x=\xi \\ 0, & x\neq\xi \end{cases}$$

$\cdot +\infty$

in 2-D.

$$\delta(x,y) = \begin{cases} \infty, & (x,y)=(0,0) \\ 0, & (x,y)\neq(0,0). \end{cases}$$

with fixed (ξ, η)

$$\delta(x-\xi, y-\eta) = \begin{cases} \infty, & (x,y)=(\xi,\eta) \\ 0, & (x,y)\neq(\xi,\eta) \end{cases}$$

- property of $\delta(x)$.

$$\int_{-\infty}^{\infty} u(x)\,\delta(x)\,dx = u(0)$$

for any continuous function $u(x)$.

$$\int_{-\infty}^{\infty} u(x)\,\delta(x-\xi)\,dx = u(\xi)$$

$$\boxed{\begin{aligned}&\int_{\mathbb{R}^2} u(x,y)\,\delta(x-\xi,\,y-\eta)\,dx\,dy\\ &= u(\xi,\eta)\\ &\text{This is also a def of } \delta .\end{aligned}}$$

(IV) property of $\Gamma(x,y)$.

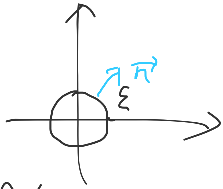
$$\Gamma = -\frac{1}{2\pi}\ln(r) = \underline{-\frac{1}{4\pi}\ln(x^2+y^2)}$$

$$\Delta\Gamma = 0 \quad \text{if } (x,y)\neq 0 .$$

$$\boxed{\Delta\Gamma \text{ at } (0,0) \quad ?}$$

$$\int_{\mathbb{R}^2} u\cdot \underline{\Delta\Gamma}\, dxdy$$

$$= \int_{B_\varepsilon} u\cdot \Delta\Gamma\, dxdy$$



$$= \underbrace{\int_{\partial B_\varepsilon} u\cdot \partial_n \Gamma\, ds}_{(i)} - \underbrace{\int_{B_\varepsilon} (\nabla u)\cdot(\nabla \Gamma)\, dxdy}_{(ii)}$$

(i): $\partial_n \Gamma = \partial_r \Gamma = -\frac{1}{2\pi r}$.

$$\int_{\partial B_\varepsilon} u\cdot\left(-\frac{1}{2\pi r}\right) ds$$

$$= \int_0^{2\pi} u\cdot \frac{-1}{2\pi r}\, r\, d\theta$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} u\, d\theta \xrightarrow{\boxed{\varepsilon\to 0}} -u(0,0)$$

average of u on ∂B_ε

(ii) $\Gamma_x = -\frac{1}{4\pi}\frac{2x}{x^2+y^2}$

$$\Rightarrow |\Gamma_x| \le C\frac{r}{r^2} \le \frac{C}{r}$$

Similarly $|\Gamma_y| \le \frac{c}{r}$

In B_ε: $\left| \int_{B_\varepsilon} (\nabla u)\cdot(\nabla\Gamma)\, dxdy \right|$

$$\le \int_{B_\varepsilon} |\nabla u| \cdot |\nabla \Gamma|\, dxdy.$$

$$\le \int_{B_\varepsilon} |\nabla u| \frac{c}{r}\, dxdy$$

$\boxed{|\nabla u| \le C_1}$

$$\le C_1 \cdot C \int_{B_\varepsilon} \frac{1}{r}\, \underline{dxdy}$$

$$= C_1 C \int_0^{2\pi} \int_0^{\varepsilon} \frac{1}{r}\, \underline{r dr d\theta}.$$

$$= 2\pi C_1 C \varepsilon \to 0$$

as $\varepsilon \to 0$.

Summary:

$$\int_{R^2} u \cdot \nabla\Gamma\, dxdy = -u(0,0)$$

$$\Rightarrow \Delta\Gamma = -\delta(x,y)$$

Note: $\Delta\Gamma = -\delta$ is def of fundamental solution.

- fundamental solution.
  at (ξ, η)

$$\Gamma(x-\xi, y-\eta)$$
$$= -\frac{1}{4\pi} \ln\left[(x-\xi)^2+(y-\eta)^2\right]$$

$$\Delta\Gamma(x-\xi, y-\eta) = -\delta(x-\xi, y-\eta)$$

(V) Green's representation formula.

for $\Delta u = f$

recall:

(1) $$\int_D v\Delta u \, dxdy = \int_{\partial D} v \cdot \partial_n u \, ds - \underline{\int_D (\nabla u)\cdot(\nabla v)\, dxdy}$$

(2) $$\int_D u \cdot \Delta v \, dxdy = \int_{\partial D} u \cdot \partial_n v \, ds$$

$$-\underline{\int_D (\nabla u)\cdot(\nabla v)\,dx\,dy}$$

(2)−(1)

$$\Rightarrow \int_D u\cdot\Delta v\,dx\,dy$$

$$= \int_{\partial D}(u\,\partial_n v - v\,\partial_n u)\,ds$$

$$- \int_D v\cdot\Delta u\,dx\,dy. \qquad (*)$$

Let $v = \Gamma(x-\xi,\, y-\eta)$

$(\xi,\eta)\in D$

$\Delta v = -\delta(x-\xi,\, y-\eta)$.

substitute into $(*)$.

$$LHS = \int_D u\cdot\Delta v\,dx\,dy$$

$$= \int_D u\cdot(-1)\cdot\delta(x-\xi,\, y-\eta)\,dx\,dy$$

$$= -\,u(\xi,\eta).$$

$(*)\Rightarrow$

$$u(\xi,\eta) = -\int(u\,\partial_n v - v\,\partial_n u)\,ds$$

∂D

$$-\int_D V \cdot \Delta u \, dx\,dy.$$

$$= \int_{\partial D} \left[-\underline{u} \cdot \partial_n \Gamma(x-\xi, y-\eta) + \Gamma(x-\xi, y-\eta)\, \underline{\partial_n u} \right] ds$$

$$- \int_D \Gamma(x-\xi, y-\eta) \cdot \underline{f} \, dx\,dy.$$

Green's representation formula.

RK1: If $\Delta u = 0$, i.e, $f = 0$

then. $u(\xi, \eta) = \int_{\partial D} \Gamma(x-\xi, y-\eta)\, \partial_n u$
$- u \,\partial_n \Gamma(x-\xi, y-\eta)\, ds$

RK2: For Dirichlet BC:

$u = g$ on ∂D.

but $\partial_n u$ is unknown

For Neumann BC.

$\partial_n u = g$ on ∂D

but u is unknown

To do list

HW13

Next: §8.2 – §8.3.