Lecture15

Friday, May 22, 2020 9:23 AM

§8.2 – §8.3

Objective: Green's function [Dirichlet problem]; Neumann function [Neumann Problem]

(I) Green function

recall: fundamental solution

$$\Gamma(x-\xi, y-\eta) = -\frac{1}{4\pi}\ln\left[(x-\xi)^2+(y-\eta)^2\right]$$

$$\Delta\Gamma = -\delta(x-\xi, y-\eta)$$

For $\Delta u = f$

Green's representation formula:

$$u(\xi,\eta) = \int_{\partial D}\left[\Gamma(x-\xi,y-\eta)\cdot\partial_n u - u\cdot\partial_n\Gamma(x-\xi,y-\eta)\right]ds - \int_D \Gamma(x-\xi,y-\eta)f(x,y)\,dx\,dy \quad (*)$$

Dirichlet problem (**)
$$\begin{cases}\Delta u = f & \text{in } D\\ u = g & \text{on } \partial D\end{cases}$$

we want $\partial_n u$ to disappear in (*)

Def: Green's function

$$\begin{cases}\Delta G(x,y,\xi,\eta) = -\delta(x-\xi,y-\eta) & \text{in } D\\ G = 0, & \text{on } \partial D\end{cases}$$

where $(\xi,\eta)\in D$ are parameters.

replace Γ by G in (*)

$$\Rightarrow u(\xi,\eta) = \int_{\partial D} G\cdot\partial_n u - u\,\partial_n G\,ds - \int_D G\cdot f\,dx\,dy$$

$\Rightarrow$ general solution of (**)

$$u(\xi,\eta) = -\int_{\partial D}\partial_n G(x,y,\xi,\eta)\cdot g(x,y)\,ds - \int_D G(x,y,\xi,\eta)f(x,y)\,dx\,dy$$

RK: $K = \partial_n G(x,y,\xi,\eta)$ is called the Poisson's kernel.

[see Poisson's formula for a disk]

(II) Properties of G

(a) The existence of G

Let $G = \Gamma + h$

then $h = G - \Gamma$

$$\begin{cases}\Delta h = \Delta G - \Delta\Gamma = 0, & \text{in } D\\ h = G - \Gamma = -\Gamma, & \text{on } \partial D\end{cases}$$

RK: Γ is known on ∂D

Thm: For smooth domain D, there exists a unique solution for h, hence a unique G.

(b) symmetry of G

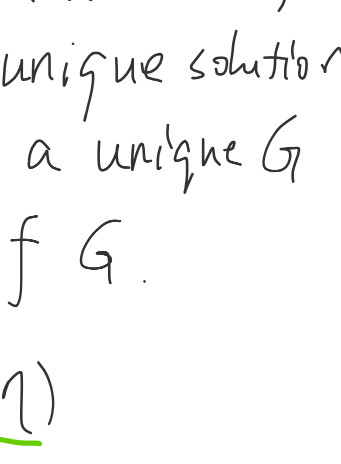
Thm: $G(x,y,\xi,\eta) = G(\xi,\eta,x,y)$ for $(x,y)\in D$, $(\xi,\eta)\in D$, $(x,y)\neq(\xi,\eta)$.

Physics:

$\Delta G = -\delta(x-\xi,y-\eta)$

G is the electric potential with point charge at (ξ,η)

electric potential at (x,y) with point charge at (ξ,η) = that at (ξ,η) with point charge at (x,y)



(c) positivity of G

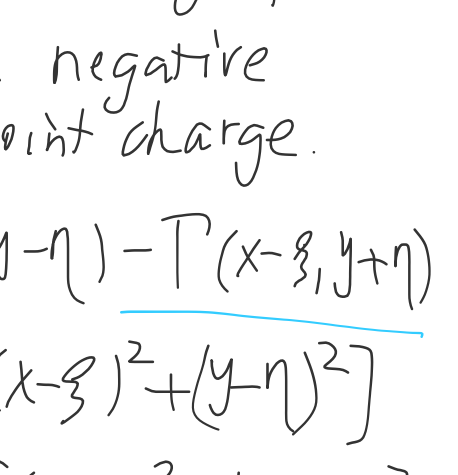
$G = \Gamma + h$,

$\Gamma = -\frac{1}{4\pi}\ln\left[(x-\xi)^2+(y-\eta)^2\right] \to +\infty$ at (ξ,η)

Thm: $G(x,y,\xi,\eta) > 0$ in $D\setminus\{\xi,\eta\}$ & $G = 0$ on ∂D

(III) construction of G

① $D = \mathbb{R}^2_+$, $y > 0$, $-\infty < x < \infty$



$$\begin{cases}\Delta G = -\delta(x-\xi,y-\eta), & \text{in } D \quad \text{(PDE)}\\ G = 0 & \text{on } y=0 \quad \text{(BC)}\end{cases}$$

use reflection principle.

$(\xi,-\eta)$ is the image point; put a negative point charge.

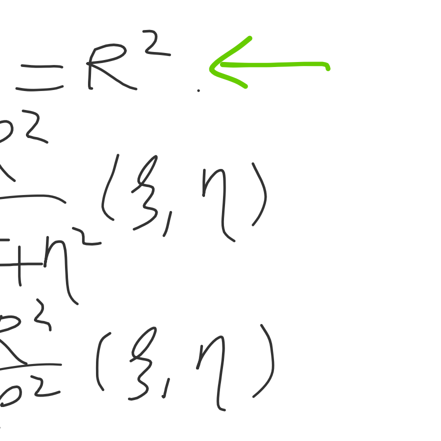
$$G = \Gamma(x-\xi,y-\eta) - \Gamma(x-\xi,y+\eta)$$
$$= -\frac{1}{4\pi}\ln\left[(x-\xi)^2+(y-\eta)^2\right] + \frac{1}{4\pi}\ln\left[(x-\xi)^2+(y+\eta)^2\right]$$

check BC: [$y=0$]

$$G = -\frac{1}{4\pi}\ln\left[(x-\xi)^2+\eta^2\right] + \frac{1}{4\pi}\ln\left[(x-\xi)^2+\eta^2\right] = 0$$

[HW: check PDE, find $K = \partial_n G$]

② A disk.



$$\begin{cases}\Delta G = -\delta(x-\xi,y-\eta) & \text{in } D\\ G = 0 & \text{on } r=R\end{cases}$$

reflection principle:

define inverse point as $Q = (\tilde\xi,\tilde\eta)$ such that

$$\overline{OP}\cdot\overline{OQ} = R^2$$

$$\Leftrightarrow (\tilde\xi,\tilde\eta) = \frac{R^2}{\xi^2+\eta^2}(\xi,\eta) = \frac{R^2}{\rho^2}(\xi,\eta)$$

where $\rho = \sqrt{\xi^2+\eta^2}$.

RK: $(\tilde\xi,\tilde\eta)\cdot(\xi,\eta) = R^2$

$$G = \begin{cases}\Gamma(x-\xi,y-\eta) - \Gamma(x-\tilde\xi,y-\tilde\eta) + \frac{1}{2\pi}\ln\left(\frac{\rho}{R}\right) & \text{if } (\xi,\eta)\neq(0,0)\\ \Gamma(x,y) + \frac{1}{2\pi}\ln(R) & \text{if } (\xi,\eta)=(0,0)\end{cases}$$

verify for case $(\xi,\eta) = (0,0)$.

$$\Delta G = \Delta\Gamma(x,y) + \Delta\left(\frac{1}{2\pi}\ln(R)\right) = -\delta(x,y) + 0 = -\delta(x,y) \quad ✓$$

at $r=R$,

$$G = \Gamma(x,y) + \frac{1}{2\pi}\ln(R) = -\frac{1}{2\pi}\ln(r) + \frac{1}{2\pi}\ln(R) = 0 \quad ✓$$

(IV) Neumann problem

(***)
$$\begin{cases}\Delta u = f & \text{in } D\\ \partial_n u = g & \text{on } \partial D\end{cases}$$

with smooth bounded D.

& $\int_D f\,dx\,dy = \int_{\partial D} g\,ds$

It is unique up to a constant.

Neumann function. N

$$\begin{cases}\Delta N = -\delta(x-\xi,y-\eta) & \text{in } D\\ \partial_n N = C_0 = \frac{-1}{L} & \text{on } \partial D\end{cases}$$

[RK: $\begin{cases}\Delta N = -\delta\\ \partial_n N = 0\end{cases}$ has no solution.

$-1 = \int_D -\delta\,dx\,dy \neq \int_{\partial D} 0\,ds = 0$]

[Necessary condition:

$-1 = \int_D -\delta\,dx\,dy = \int_{\partial D} C_0\,ds = C_0\cdot L$

where L is length of ∂D.

$\Rightarrow C_0 = \frac{-1}{L}$]

replace Γ by N in (*) of (I)

$\Rightarrow$ general solution

$$u(\xi,\eta) = \int_{\partial D} N(x,y,\xi,\eta)\,g(x,y)\,ds - \int_D N(x,y,\xi,\eta)f(x,y)\,dx\,dy + \frac{1}{L}\int_{\partial D} u\,ds$$

To do list

HW: Ex8.5

[The solution is given in textbook, show steps or verify it]