Lecture16

Wednesday, May 27, 2020 9:23 AM

§9.2

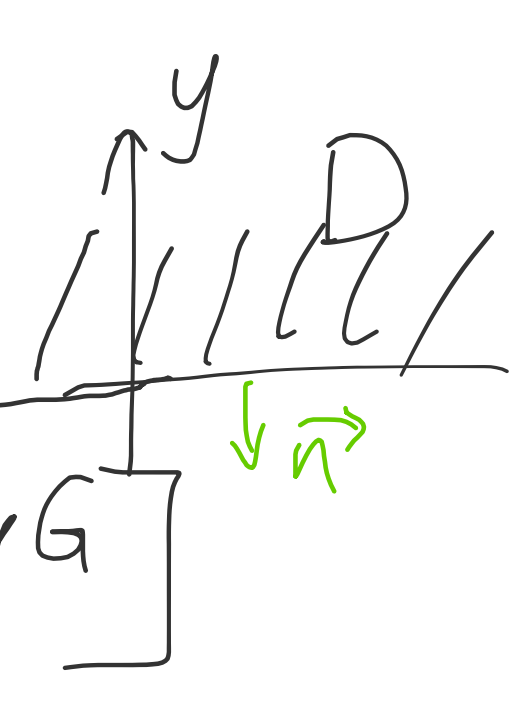
Objective: review: ch 8
First-order Equations
(high dimension)

Review:

$$\begin{cases}\Delta u=f & \text{in } D\\ u=g & \partial D\end{cases}$$

$$u=-\int_D G\cdot f\,dxdy-\int_{\partial D}\partial_n G\cdot g\,ds$$
$$=-\int_D G\cdot f\,dxdy+\int_{\partial D}K\cdot g\,ds$$

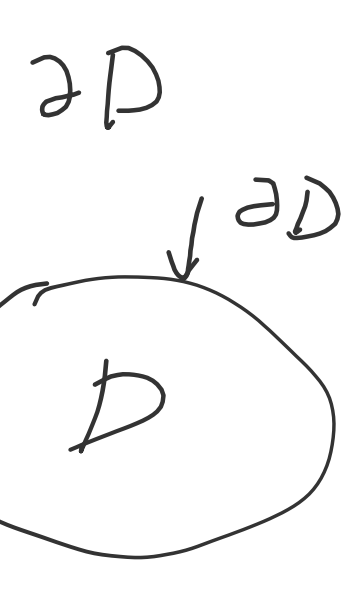
where
$K=-\partial_n G$



$$\left[\text{in HW, } K=-\partial_n G=\partial_y G\right]$$

$$(*)\ \begin{cases}\Delta u=f & \text{in } D\\ \partial_n u=g & \text{on } \partial D\end{cases}$$

$$\begin{cases}\Delta N=-\delta(x-\xi,y-\eta), & \text{in } D\\ \partial_n N=\frac{1}{L}, & \text{on } \partial D\end{cases}$$



general solution of $(*)$

$$u(\xi,\eta)=\int_{\partial D}N(x,y,\xi,\eta)\,g(x,y)\,ds$$
$$-\int_D N(x,y,\xi,\eta)\,f(x,y)\,dxdy$$
$$+\frac{1}{L}\int_{\partial D}u\,ds$$

RK1: $N$ is not unique
we select $N$ such that
$\int_{\partial D}N\,ds=0$
(normalization)

RK2 $\int_{\partial D}u\,ds$ is a constant,
$u$ is unique up to a constant,
So we set $\int_{\partial D}u\,ds=0$
(normalization)

So $u(\xi,\eta)=\int_{\partial D}N\cdot g\,ds$
$-\int_D N\cdot f\,dxdy$
is one solution

## (I) Quasilinear equations and CE

recall: in 2-D case
$a(x,y,u)\,u_x+b(x,y,u)\,u_y=c(x,y,u)$

$$(CE)_2\ \begin{cases}x_t=a\\ y_t=b\\ u_t=c\end{cases}$$

In n-dimension (n-D) case

$$\sum_{i=1}^{n}a_i(x_1,x_2,\cdots,x_n,u)\,u_{x_i}$$
$$=c(x_1,x_2,\cdots,x_n,u)$$

$u(x_1,x_2,\cdots,x_n)$

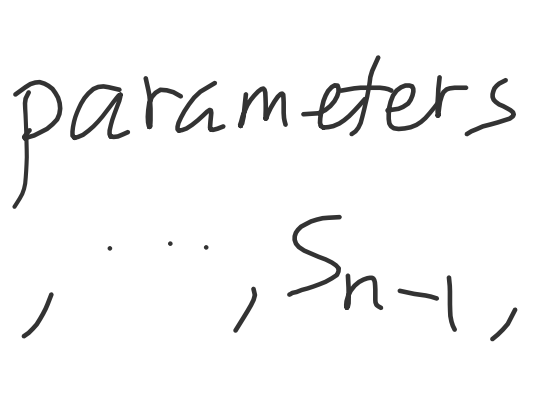
$$(CE)_n:\ \begin{cases}\frac{\partial x_i}{\partial t}=a_i, & i=1,\cdots,n\\ \frac{\partial u}{\partial t}=c\end{cases}$$

Proof: set $\frac{\partial x_i}{\partial t}=a_i$

then $\frac{\partial u}{\partial t}\overset{\text{chain rule}}{=}\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial t}\overset{\text{Eq}}{=}c$

## (II) Initial condition Γ

recall: Γ is curve in 2D
use parameter s



$$(IC)_2\ \begin{cases}x(0,s)=x_0(s)\\ y(0,s)=y_0(s)\\ u(0,s)=u_0(s)\end{cases}$$

In general case (n-D)
Γ is (n-1)-D surface in $\mathbb{R}^{n+1}$ space

we use n-1 parameters
$s_1,s_2,\cdots,s_{n-1}$,

At t=0

$$(IC)_n\ \begin{cases}x_i(0,s_1,\cdots,s_{n-1})=x_{i,0}(s_1,\cdots,s_{n-1})\\ \quad(i=1,\cdots,n)\\ u(0,s_1,\cdots,s_{n-1})=u_0(s_1,\cdots,s_{n-1})\end{cases}$$

Summary:

step1: $(CE)_n$ & $(IC)_n$

step2: solve
$\rightarrow x_i(t,s_1,\cdots,s_{n-1})$
$u(t,s_1,\cdots,s_{n-1})$

step3: $(t,s_1,s_2,\cdots,s_{n-1})$
$\longleftrightarrow(x_1,x_2,\cdots,x_n)$

## (IV) Existence & Uniqueness

Jacobian J on Γ

recall: 2D

$$J|_\Gamma=\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s}\\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t}\end{vmatrix}_{t=0}$$
$$=\begin{vmatrix}(x_0)_s, & (y_0)_s\\ a, & b\end{vmatrix}_{t=0}$$

If $J|_\Gamma\neq0$, ⇒ unique solution

If $J|_\Gamma=0$ ⇒ $\begin{cases}\text{no solution}\\ \infty\text{ many solutions}\end{cases}$

In n-D case

$$J|_\Gamma=\begin{vmatrix}\frac{\partial x_1}{\partial s_1} & \frac{\partial x_2}{\partial s_1}, & \cdots & \frac{\partial x_n}{\partial s_1}\\ \vdots & & & \\ \frac{\partial x_1}{\partial s_{n-1}}, & \frac{\partial x_2}{\partial s_{n-1}}, & \cdots & \frac{\partial x_n}{\partial s_{n-1}}\\ \frac{\partial x_1}{\partial t}, & \frac{\partial x_2}{\partial t}, & \cdots, & \frac{\partial x_n}{\partial t}\end{vmatrix}_{t=0}$$

$$=\begin{vmatrix}(x_{1,0})_{s_1}, & \cdots & , (x_{n,0})_{s_1}\\ \vdots & & \\ (x_{1,0})_{s_{n-1}}, & \cdots, & (x_{n,0})_{s_{n-1}}\\ a_1 & \cdots\cdots & a_n\end{vmatrix}_{t=0}$$

Similarly:

If $J|_\Gamma\neq0$, ⇒ unique solution

If $J|_\Gamma=0$ ⇒ $\begin{cases}\text{no solution}\\ \infty\text{ many solutions}\end{cases}$

## (V) Examples

Solve: $x\,u_x+y\,u_y+z\,u_z=4u$
with (IC): $u(x,y,1)=xy$

step1:

$$CE:\ \begin{cases}x_t=x & (1)\\ y_t=y & (2)\\ z_t=z & (3)\\ u_t=4u & (4)\end{cases}$$

(IC): $u(x,y,1)=xy$
$\Longleftrightarrow u(\underline{s_1},\underline{s_2},\underline{1})=\underline{s_1 s_2}$

at t=0

$$(IC)\ \begin{cases}x(0,s_1,s_2)=s_1 & (5)\\ y(0,s_1,s_2)=s_2 & (6)\\ z(0,s_1,s_2)=1, & (7)\\ u(0,s_1,s_2)=s_1 s_2, & (8)\end{cases}$$

step2:

(1) ⇒ $x=C_1(s_1,s_2)e^t$

(5) ⇒ $x(0,s_1,s_2)=C_1(s_1,s_2)\overset{(5)}{=}s_1$

$$\boxed{x=s_1e^t}$$

(2)(6) ⇒ $y=s_2e^t$ [Exercise]

(3)(7) ⇒ $z=e^t$

(4)(8) ⇒ $\begin{cases}u=C_4\,e^{4t}\\ u(0,s_1,s_2)=C_4=s_1s_2\end{cases}$

$u=s_1s_2e^{4t}$

step3: convert $u(t,s_1,s_2)$
to $u(x,y,z)$

$e^t=z$ (9)

$x=s_1e^t\Rightarrow s_1=\frac{x}{e^t}\overset{(9)}{=}\frac{x}{z}$

$y=s_2e^t\Rightarrow s_2=\frac{y}{e^t}\overset{(9)}{=}\frac{y}{z}$

$u=s_1s_2e^{4t}$
$=\frac{x}{z}\,\frac{y}{z}\,(e^t)^4$
$\overset{(9)}{=}\frac{x}{z}\,\frac{y}{z}\,z^4=xyz^2$

RK:

$$J|_\Gamma=\begin{vmatrix}(s_1)_{s_1}, & (s_2)_{s_1}, & (1)_{s_1}\\ (s_1)_{s_2} & (s_2)_{s_2} & (1)_{s_2}\\ x & y & z\end{vmatrix}_{t=0}$$
$$=\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ s_1 & s_2 & 1\end{vmatrix}$$

$=1\neq0$

⇒ a unique solution.

② Solve
$2u_x+(u_y+u_z)\tan u=0$
IC: $u(1,y,z)=\arctan(y+z)$

step1:

$$CE:\ \begin{cases}x_t=2 & (1)\\ y_t=\tan(u) & (2)\\ z_t=\tan(u) & (3)\\ u_t=0 & (4)\end{cases}$$

(IC): $u(\underline{1},\underline{s_1},\underline{s_2})=\underline{\arctan(s_1+s_2)}$

At t=0

$$\begin{cases}x(0,s_1,s_2)=1 & (5)\\ y(0,s_1,s_2)=s_1 & (6)\\ z(0,s_1,s_2)=s_2 & (7)\\ u(0,s_1,s_2)=\arctan(s_1+s_2) & (8)\end{cases}$$

step2:

$$\left.\begin{array}{l}(1)\Rightarrow x=2t+C_1\\ (5)\Rightarrow 0+C_1=1\end{array}\right\}$$
$\Rightarrow x=2t+1$

$$\left.\begin{array}{l}(4)\Rightarrow u=C_2(s_1,s_2)\\ (8)\Rightarrow C_2=\arctan(s_1+s_2)\end{array}\right\}$$
$\Rightarrow u=\arctan(s_1+s_2)$

[Exercise]
[Final answer $u=\arctan\left(\frac{y+z}{x}\right)$]

To do list

HW

Next §9.4