Lecture17

Friday, May 29, 2020 9:19 AM

§9.4

Objective $\begin{cases} \text{review: 1st-order PDE} \\ \text{wave equation in } \mathbb{R} \text{ \& } \mathbb{R}^3 \\ \quad [\,u(x,t),\ u(r,t)\,] \end{cases}$

(I) Review

Solve $2u_x + (u_y + u_z)\tan(u) = 0$

with: $u(1,y,z) = \arctan(y+z)$

Solution:

Step 1:

$$\text{CE} \begin{cases} x_t = 2, & (1) \\ y_t = \tan(u), & (2) \\ z_t = \tan(u), & (3) \\ u_t = 0, & (4) \end{cases}$$

$$\begin{matrix} \text{IC:} \\ t=0 \end{matrix} \begin{cases} x(0,s_1,s_2) = 1 & (5) \\ y(0,s_1,s_2) = s_1 & (6) \\ z(0,s_1,s_2) = s_2 & (7) \end{cases}$$

$$| \, u(0, s_1, s_2) = \arctan(s_1+s_2) \quad (8)$$

step2:

[lecture 16]

$$x = 2t+1$$

$$u = \arctan(s_1+s_2) \quad (*)$$

substitute $(*)$ into (2).

$$y_t = \tan(u) \overset{(*)}{=} s_1+s_2$$

$$\left.\begin{aligned} &\Rightarrow \quad y = (s_1+s_2)t + C_3 \\ &(6) \Rightarrow \quad y(0,s_1,s_2) = C_3 \overset{(6)}{=} s_1 \end{aligned}\right\}$$

$$y = \underline{(s_1+s_2)}t + \underline{s_1} \quad \leftarrow$$

substitute $(*)$ into (3)

$$z_t = \tan(u) = s_1+s_2$$

$$\left.\begin{aligned} &\Rightarrow z = (s_1+s_2)t + C_4 \\ &(7) \Rightarrow z(0,s_1,s_2) = C_4 = s_2 \end{aligned}\right\}$$

$$z = \underline{(s_1+s_2)}t + \underline{s_2} \quad \leftarrow$$

step3: $u = \arctan(s_1+s_2)$

$$y+z=(s_1+s_2)2t+\underline{s_1+s_2}$$

$$=(s_1+s_2)(2t+1)$$

$$=(s_1+s_2)\,x$$

$$s_1+s_2=\frac{y+z}{x}$$

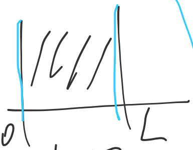
$$u=\arctan\left(\frac{y+z}{x}\right)$$

RK: check $J|_{\Gamma}\neq 0$ [Exercise]

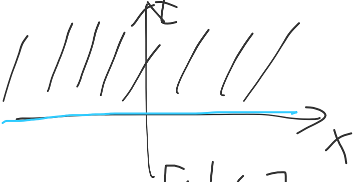
(II) 1-D wave equation [Ch4]

Cauchy problem: $x\in\mathbb{R}$

$$\begin{cases} u_{tt}=c^2u_{xx} & x\in\mathbb{R},\ t>0 \quad \text{(PDE)} \\ u(x,0)=f(x), & x\in\mathbb{R} \\ u_t(x,0)=g(x), & x\in\mathbb{R} \end{cases} \Big\} \text{IC}$$



[ch5]



[ch4]

main results:

① $u = F(x-ct) + G(x+ct)$

(By PDE)

② $F, G \leftarrow f, g$

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

D'Alembert's formula.

[For Exercises,
see Example 4.2, & 4.3
in ch4]

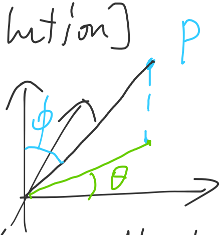
(III) 3-D wave Equation

$u = u(\vec{x}, t)$

$\vec{x} = (x_1, x_2, x_3)$

For radial symmetric solution
[radial solution]

$u = u(r, t)$



Note! spherical coordinates

$$\begin{cases} x_1 = r\sin\phi\cos\theta \\ x_2 = r\sin\phi\sin\theta \\ x_3 = r\cos\phi \end{cases}$$

3-D wave Equation

$u_{tt} = c^2 \Delta u$ for $u(\vec{x}, t)$

For $u(r,t)$

$$\begin{cases} u_{tt} = c^2\left[u_{rr} + \frac{2}{r}u_r\right] & \text{(PDE)} \\ \qquad 0 \le r < \infty,\ t > 0 \\ u(r,0) = f(r) & 0 \le r < \infty \quad \text{(IC)} \\ u_t(r,0) = g(r) \end{cases}$$

Step 1: convert PDE to 1-D wave Eq.

Let $\underline{V(r,t)} = r \cdot u(r,t)$

$$\Rightarrow \begin{cases} V_{tt} = c^2 V_{rr}, & \text{[Exercise]} \\ V(r,0) = \underline{r f(r)} & 0 \le r < \infty \\ V_t(r,0) = \underline{r g(r)} \end{cases}$$

By 1-D wave Eq in (I),
we formally get

$$V = \frac{(r+ct)f(r+ct) + (r-ct)f(r-ct)}{2} + \frac{1}{2c}\int_{r-ct}^{r+ct} s\, g(s)\, ds \qquad (*)$$

One Problem: f, g are defined only for $r \geq 0$

step 2: Extend f, g to $-\infty < r < \infty$.

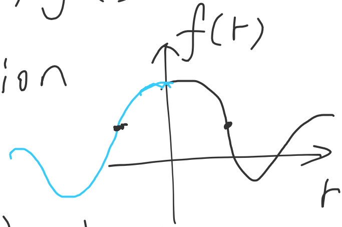
For a radial solution

$u = u(r)$

symmetry $\Rightarrow u'(0) = 0$

So $f'(0) = 0$, $g'(0) = 0$

Even extension of f, g.



$$\tilde{f}(r) = \begin{cases} f(r), & r \geq 0 \\ f(-r), & r < 0 \end{cases}$$

$$\tilde{g}(r) = \begin{cases} g(r), & r \geq 0 \\ g(-r), & r < 0 \end{cases}$$

replace f, g by $\tilde{f}, \tilde{g}$ in $(*)$

use $\underline{V = r u}$

general solution

$$\Rightarrow \boxed{u = \frac{V}{r} = \frac{1}{2r}\Big[(r+ct)\tilde{f}(r+ct) + (r-ct)\tilde{f}(r-ct)\Big] + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\,\tilde{g}(s)\,ds}$$

(IV) Examples:

① $$\begin{cases} u_{tt} = \Delta u, & 0 \le r < \infty,\ t > 0 \\ u(r,0) = \underline{0} & 0 \le r < \infty \\ u_t(r,0) = \begin{cases} 1, & 0 \le r \le 1 \\ 0, & r > 1 \end{cases} \end{cases}$$

$u = u(r,t)$

Find $u(2, \frac{1}{2})$, $u(2, \frac{3}{2})$, $u(2, 4)$.

Solution:

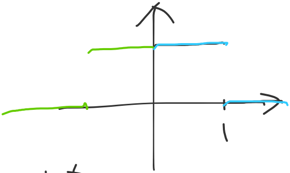
$$f = 0 \Rightarrow \tilde{f} = 0$$

$$g(r) = \begin{cases} 1, & 0 \le r \le 1 \\ 0, & r > 1 \end{cases}$$

$$\tilde{g} = \begin{cases} 1, & -1 \leq r \leq 1 \\ 0, & r > 1, \text{ or } r < -1 \end{cases}$$

$c^2 = 1$

$c = 1$.



$$u(r,t) = \frac{1}{2r}\int_{r-t}^{r+t} s\,\tilde{g}(s)\,ds$$

So: $$u(2,\tfrac{1}{2}) = \frac{1}{4}\int_{3/2}^{5/2} s\,\tilde{g}(s)\,ds$$

$$= 0$$

$$u(2,\tfrac{3}{2}) = \frac{1}{4}\int_{\frac{1}{2}}^{7/2} s\,\tilde{g}(s)\,ds$$

$$= \frac{1}{4}\int_{\frac{1}{2}}^{1} s\,\tilde{g}(s)\,ds + \frac{1}{4}\int_{1}^{7/2} s\,\tilde{g}(s)\,ds$$

$$= \frac{1}{4}\int_{\frac{1}{2}}^{1} s\,ds + 0$$

$$= \frac{1}{4} \frac{s^2}{2} \Big|_{\frac{1}{2}}^{1}$$

$$= \frac{1}{8}\left[1 - \frac{1}{4}\right] = \frac{3}{32}$$

$$u(2,4) = \frac{1}{4} \int_{-2}^{6} s\,\tilde{g}(s)\,ds$$

$$= \frac{1}{4} \int_{-1}^{1} s\,\tilde{g}(s)\,ds$$

$$= \frac{1}{4} \int_{-1}^{1} s\,ds = 0$$

② For $u = u(r,t)$ in $\mathbb{R}^3$

$$\begin{cases} u_{tt} = \underline{4}\Delta u \\ u(r,0) = f(r) = \begin{cases} 1, & 0 \le r \le 1 \\ 0, & r > 1 \end{cases} \\ u_t(r,0) = 0 \end{cases}$$

Find $u(r,t)$ for some fixed $r > 1$, and all $t > 0$.

Solution: $c^2 = 4 \Rightarrow c = 2$

$$g(r) = 0 \Rightarrow \tilde{g}(r) = 0$$

$$\tilde{f}(r) = \begin{cases} 1, & -1 \le r \le 1 \\ 0, & r > 1, \text{ or } r < -1. \end{cases}$$

$$u(r,t) = \frac{1}{2r}\Big[(r+2t)\tilde{f}(\underline{r+2t}) + (r-2t)\tilde{f}(r-2t)\Big]$$

$$r>1,\ t>0 \Rightarrow r+2t>1$$

$$\Rightarrow \tilde{f}(r+2t) = 0$$

So $u(r,t) = \dfrac{r-2t}{2r}\tilde{f}(r-2t).$

$$u(r,t) = \begin{cases} \dfrac{r-2t}{2r}, & \text{if } -1 \le r-2t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$-1 \le r-2t \le 1$$

$$\Leftrightarrow -1-r \le -2t \le 1-r$$

$$\Leftrightarrow \frac{r-1}{2} \le t \le \frac{1+r}{2}$$

$[\text{Note } r>1]$

$$u(r,t)=\begin{cases}\dfrac{r-2t}{2r}, & \dfrac{r-1}{2}<t\le\dfrac{1+r}{2}\\ 0, & \text{otherwise}\end{cases}$$

$$0\le t<\frac{r-1}{2} \text{ or } t>\frac{1+r}{2}$$

To do list

HW16

Next §9.4.2