Lecture10

Wednesday, May 6, 2020 9:25 AM

§7.2 – §7.3

Objective { review
Existence of solution
maximum principle
(uniqueness)

(I) Review

in 2D: $\Delta u = u_{xx} + u_{yy}$

if $u_{xx} + u_{yy} = 0$ Laplace Eq.
u is called harmonic function

e.g.: $P_1 = x - y$, $\Delta P_1 = 0$.

$P_2 = x^2 - y^2$

$(P_2)_{xx} = 2 + 0 = 2$

$(P_2)_{yy} = 0 - 2 = -2$

$\Delta P_2 = (P_2)_{xx} + (P_2)_{yy} = 0$

In Polar Coordinates

$u(x,y) = w(r,\theta)$

$\Delta u = \Delta w = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} = 0$

if $w(r,\theta) = w(r)$

$\Rightarrow w_{rr} + \frac{1}{r} w_r = 0$.

one harmonic function

$w(r) = -\frac{1}{2\pi} \ln(r)$

[fundamental solution]

D, ∂D

BC:

Dirichlet: $u = g$ on ∂D

Neumann: $\partial_n u = g$ on ∂D

Robin: $u + \alpha \partial_n u = g$, on ∂D.

(II): Existence of solution

① $\begin{cases} \Delta u = F, \text{ in } D, & \text{Poisson's Eq} \\ u = g, & \partial D \end{cases}$

Thm: There exist solutions for ① if D is bounded & smooth.

[constructed by separation of variables]

② $\begin{cases} \Delta u = F & \text{in } D \\ \partial_n u = g & \text{on } \partial D. \end{cases}$

Thm: A necessary condition for existence is

$\int_{\partial D} g(x(s), y(s))\, ds = \int_D F(x,y)\, dx dy$.

Proof: Gauss's formula.

$\int_D \nabla \cdot \vec{f}\, dx dy = \int_{\partial D} \vec{f} \cdot \vec{n}\, ds$

PDE: $F = \Delta u = \nabla \cdot (\nabla u)$

integrate: $\int_D F\, dx dy = \int_D \nabla \cdot (\underbrace{\nabla u}_{\vec{f}})\, dx dy$

$\vec{f} = \nabla u = \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}$

$\overset{\text{Gauss}}{=} \int_{\partial D} (\nabla u) \cdot \vec{n}\, ds$

$= \int_{\partial D} \partial_n u\, ds$

$\overset{BC}{=} \int_{\partial D} g\, ds$.

RK: Laplace Eq: $\Delta u = 0$

$F = 0$, $\rightarrow \int_{\partial D} g\, ds = 0$

(III) Maximum principle

PDE system Q: { existence
uniqueness
stability }

maximum principle { weak ~
strong ~

Thm: weak maximum principle

For $\Delta u = 0$. in D

suppose $u \in C^2(D) \cap C(\bar{D})$

and D is bounded

$\Rightarrow$ maximum of u is achieved on ∂D.

[math: $\max_D u \le \max_{\partial D} u$]

RK: $\bar{D} = D \cup \partial D$

eg: D: $|r| < 1$

$\bar{D}$: $|r| \le 1$

D

Proof:

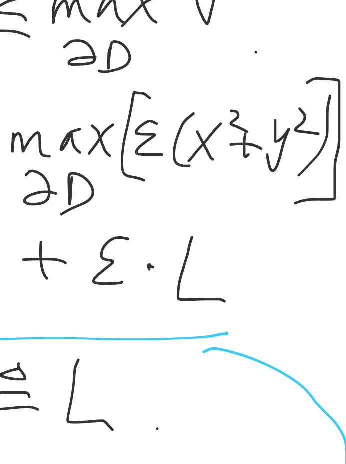
(i) Claim: if $\Delta v > 0$ in D then v can not have a local max in D.

Proof of claim

if $(x_0, y_0) \in D$ is local max

$\Rightarrow \Delta v \le 0$, at (x_0, y_0)

contradiction.



$f''(x) \le 0$

$f'(x) = 0$

(ii)

define $v = u + \varepsilon(x^2 + y^2)$, $\varepsilon > 0$

then $\Delta v = \Delta u + \varepsilon \Delta(x^2 + y^2)$

$= 0 + 4\varepsilon > 0$

By (i): $\max_D v \le \max_{\partial D} v$.

$\max_{\partial D} v = \max_{\partial D} u + \max_{\partial D} [\varepsilon(x^2 + y^2)]$

$= \max_{\partial D} u + \varepsilon \cdot L$

$\max_{\partial D} (x^2 + y^2) \triangleq L$.

L is finite, since D is bounded

$\max_D u \le \max_D v$

$\le \max_{\partial D} u + \varepsilon L$

Let $\varepsilon \to 0$, $\max_D u \le \max_{\partial D} u$.

RK: minimum of u is also achieved on ∂D.

$\Delta u = 0 \Leftrightarrow \Delta(-u) = 0$

$\max_D(-u) \le \max_{\partial D}(-u)$

[$\min_D(u) = -\max_D(-u)$]

$-\max_D(-u) \ge -\max_{\partial D}(-u)$

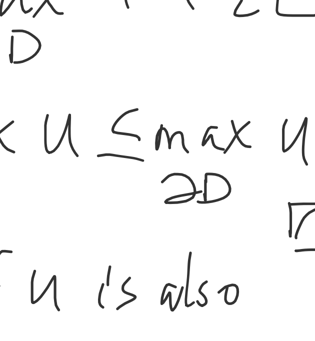
$\min_D(u) \ge \min_{\partial D}(u)$

RK: boundedness of D is essential.

counterexample:

D: $|r| > 1$

∂D: $|r| = 1$



$\Delta u = 0$, $u = \ln(x^2 + y^2)$

is a solution.

on ∂D, $u = \ln(1) = 0$

$\ln(x^2 + y^2) \to \infty$

as $r = \sqrt{x^2 + y^2} \to \infty$

Thm: mean value principle

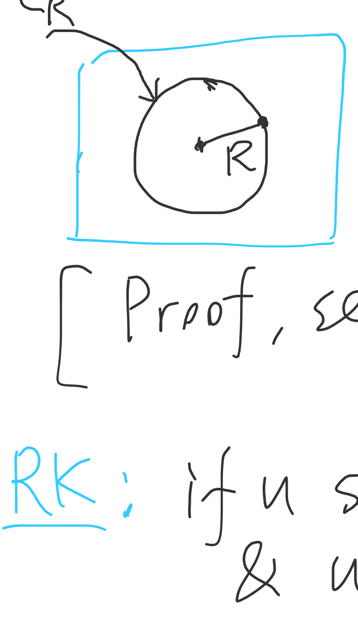
$\Delta u = 0$.

$\Rightarrow u(x_0, y_0) = \frac{1}{2\pi R} \oint_{C_R} u(x(s), y(s))\, ds$ (*)

$= \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\, d\theta$

[Note, $ds = R\, d\theta$]

C_R



[Proof, see textbook, discussion section]

RK: if u statisfies (*) & $u \in C^2(D)$

$\Rightarrow \Delta u = 0$ in D.

Thm: strong maximum principle

If $\Delta u = 0$. in D

[D is assumed to be bounded]

and if u attains its max in D

$\Rightarrow$ u is a constant in D.

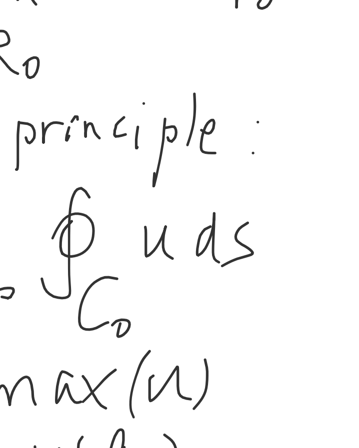
RK: math: $\max_D u \le \max_{\partial D} u$

if $\max_D(u) = \max_{\partial D}(u) \Leftrightarrow u = C$.

Proof:

$q_0 = (x_0, y_0)$

suppose $u(q_0)$ is max.



$\forall q, (x, y)$

there exists L connecting q & q_0.

define C_0 with center q_0 & radiu R_0

By mean value principle:

$u(q_0) = \frac{1}{2\pi R_0} \oint_{C_0} u\, ds$

$\le \max(u)$

$= u(q_0)$

$\Rightarrow u = u(q_0)$ on C_0.

$q_1 = C_0 \cap L$, $u(q_1) = u(q_0)$

construct $C_1 \to q_2 \to C_2 \to q_3 \cdots \to q$

$\Rightarrow u(q) = \cdots = u(q_2) = u(q_1) = u(q_0)$

So $u(q) = u(q_0)$, $\forall q$.

$\Leftrightarrow$ u is a constant.

To do list

HW9

Next: §7.4 – §7.6.