§7.4 – §7.6

Objective: review: maximum principle (MP)
applications of MP
Energy method

[I] Review

$\Delta u = 0$ in D (bounded)

weak MP: $\to \max_D(u) \le \max_{\partial D}(u)$
$\to \min_D(u) \ge \min_{\partial D}(u)$

strong MP: "=" implies $u(x,y) = c$ in D

[II] uniqueness & stability of $\Delta u = F$

Thm: Suppose D is bounded.

$$\begin{cases} \Delta u = F & \text{in } D \\ u = g & \text{on } \partial D \end{cases}$$

Then, it has at most one solution in $C^2(D)\cap C(\bar D)$

Proof. Assume there are two solutions u_1 & u_2

$$\begin{cases} \Delta u_1 = F, & \text{in } D \\ u_1 = g, & \partial D \end{cases} \quad \begin{cases} \Delta u_2 = F, & \text{in } D \\ u_2 = g, & \partial D \end{cases}$$

Let $v = u_1 - u_2$

$$\begin{cases} \Delta v = 0 & \text{in } D \\ v = 0 & \text{on } \partial D \end{cases}$$

By MP:

$v(x,y) \le \max_D(v) \le \max_{\partial D}(v) = 0$

$v(x,y) \ge \min_D(v) \ge \min_{\partial D}(v) = 0$

$\Rightarrow v = 0$ in D.

$\Rightarrow u_1 = u_2$

RK: boundedness is essential

counterexample:

$$\begin{cases} \Delta u = 0, & \text{in } D \\ u = 0, & \text{on } r = 1 \ [x^2+y^2=1] \end{cases}$$

$u_1 = 0, \quad u_2 = -\frac{1}{2\pi}\ln(r)$

$\dots \quad u_3 = \ln(x^2+y^2)$

Thm (stability) [Thm 7.13]

suppose D is bounded

$$\begin{cases} \Delta u_1 = F & \text{in } D \\ u_1 = g_1 & \text{on } \partial D \end{cases} \quad \begin{cases} \Delta u_2 = F & \text{in } D \\ u_2 = g_2 & \text{on } \partial D \end{cases}$$

then $\max_D |u_1 - u_2| \le \max_{\partial D} |g_1 - g_2|$

Proof [Exercise]

RK: MP $\to$ Dirichlet problem ✓
$\to$ Neumann problem
Robin problem ✗

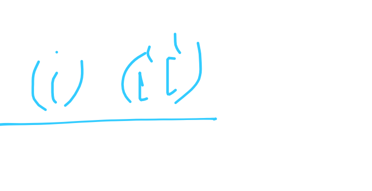
[III] Energy method (Green's identities)

recall: Gauss's formula

$\int_D \nabla\cdot\vec F\,dxdy = \int_{\partial D} \vec F\cdot\vec n\,ds$

$\vec F = \nabla u$

1-D: $\int_a^b f_x\,dx = f(b) - f(a)$



(i) $\int_D \Delta u\,dxdy = \int_{\partial D} \partial_n u\,ds$ $\leftrightarrow$ $\int_a^b u_{xx}\,dx = u_x(b) - u_x(a)$

integration by parts

(ii) $\int_D v\cdot\Delta u\,dxdy = \int_{\partial D} v\cdot\partial_n v\,ds - \int_D (\nabla u)\cdot(\nabla v)\,dxdy$ $\leftrightarrow$ $\int_a^b v\,u_{xx}\,dx = u_x\cdot v\Big|_a^b - \int_a^b u_x v_x\,dx$

Green's identities (i) (ii)

RK: By $v=1$ in (ii)

(ii) $\Rightarrow$ (i)

Thm: (uniqueness of $\Delta u = F$)

Let D be bounded.

(a) $$\begin{cases} \Delta u = F, & \text{in } D \\ u = g & \text{on } \partial D \end{cases}$$

it has at most one solution

(b) $$\begin{cases} \Delta u = F & \text{in } D \\ \partial_n u = g, & \text{on } \partial D \end{cases}$$

if u is a solution, then any solution is of the form

$W = u + C, \quad C \in \mathbb{R}$

(c) $$\begin{cases} \Delta u = F & \text{in } D \\ u + \alpha\partial_n u = g, & \text{on } \partial D. \end{cases}$$

If $\alpha \ge 0$, then it has at most one solution

Proof: Let u_1 and u_2 be two solutions.

$v = u_1 - u_2$

$\overset{PDE}{\Longrightarrow}$ $\Delta v = \Delta u_1 - \Delta u_2$
$= F - F = 0$

$\int_D v\cdot\Delta v\,dxdy = 0$

$\overset{Green's}{\Longrightarrow}$ $\int_{\partial D} v\cdot\partial_n v\,ds - \int_D (\nabla v)\cdot(\nabla v)\,dxdy = 0$

$\Rightarrow$ (*): $\int_D |\nabla v|^2\,dxdy = \int_{\partial D} v\cdot\partial_n v\,ds$

For (a): $v = u_1 - u_2 = g - g = 0$ on ∂D

By (*): $\int_D |\nabla v|^2\,dxdy = 0$

$\nabla v = \vec 0, \Rightarrow v = C$ in D

since $v = 0$ on ∂D

$v(x,y) = 0$ in D.

For (b):

$\partial_n v = \partial_n u_1 - \partial_n u_2$
$= g - g = 0$, on ∂D

By (*) $\Rightarrow \int_D |\nabla v|^2\,dxdy = 0$

$\Rightarrow \nabla v = \vec 0$

$\Rightarrow v = C$ in D

$\Rightarrow u_1 - u_2 = C$

For (c):

$v + \alpha\partial_n v = u_1 + \alpha\partial_n u_1 - (u_2 + \alpha\partial_n u_2)$
$= g - g = 0$, on ∂D

$\Rightarrow v = -\alpha\partial_n v$ on ∂D $\leftarrow$

By (*): $0 \le \int_D |\nabla v|^2\,dxdy = \int_{\partial D} v\,\partial_n v\,ds$
$= -\int_{\partial D} \alpha(\partial_n v)^2\,ds$
≤ 0

$\Rightarrow \begin{cases} \nabla v = \vec 0, & \text{in } D \\ \partial_n v = 0, & \text{on } \partial D. \end{cases}$

$\Rightarrow \begin{cases} v = C & \text{in } D \\ v = -\alpha\partial_n v = 0, & \text{on } \partial D \end{cases}$

$\Rightarrow v = 0$ in D

$\Longleftrightarrow u_1 = u_2$

Thm: stability of heat Equation

Let $u_t = k\Delta u$, (1D, 2D, 3D) (x,y,z)

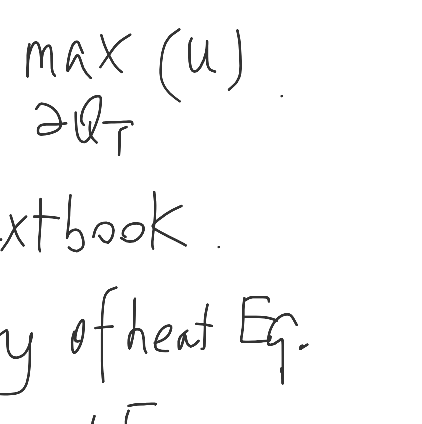
in $Q_T = D \times [0,T]$
$= \{(x,y,z)\in D, t \in [0,T]\}$

$\partial Q_T = \{D\times\{0\}\} \cup \{\partial D \times [0,T]\}$

for 1D space

$x \in [0, L]$

$t \in [0, T)$



$Q_T = [0,L]\times[0,T]$
$= \{x \in [0,L], t \in [0,T]\}$

Thm: maximum principle

If $u_t = k\Delta u$, in Q_T

then $\max_{Q_T}(u) \le \max_{\partial Q_T}(u)$.

Proof see textbook.

Thm: stability of heat Eq.

Let $$\begin{cases} (u_1)_t = k\Delta u_1 + F \\ u_1 = g_1 & \text{on } \partial D \times [0,T] \\ u_1 = f_1 & \text{at } t = 0 \end{cases}$$

$$\begin{cases} (u_2)_t = k\Delta u_2 + F \\ u_2 = g_2 & \text{on } \partial D \times [0,T] \\ u_2 = f_2 & \text{at } t = 0 \end{cases}$$

$D\times\{0\}$

Then:

$\max_{Q_T} |u_1 - u_2| \le \max_D |f_1 - f_2| + \max_{\partial D\times[0,T]} |g_1 - g_2|$

Proof: $v = u_1 - u_2$
& use MP for heat Eq.

RK: it implies uniqueness by $f_1 = f_2$, $g_1 = g_2$

To do list.
HW10.
Next, §7.7