1.

(a) $\boxed{4pts}$ 2nd-order $\nearrow$ 2pts linear. PDE. $\nearrow$ 2pts

(b) $\boxed{3pts}$ $r_x = \frac{x}{r}$, $r_y = \frac{y}{r}$

$$u_x = h'(r)\, r_x = h'(r)\frac{x}{r}$$

$$u_y = h'(r)\, r_y = h'(r)\frac{y}{r} \longrightarrow 1pt$$

$$u_{xx} = \left(\frac{h'(r)}{r}\cdot x\right)_x$$

$$= \frac{h'(r)}{r} + \frac{h''(r)\cdot r - h'(r)}{r^2}\cdot r_x \cdot x$$

$$= \frac{h'(r)}{r} + \frac{h''(r)\cdot r - h'(r)}{r^2}\cdot\frac{x^2}{r}$$

$$u_{yy} = \left(\frac{h'(r)}{r}\cdot y\right)_y$$

$$= \frac{h'(r)}{r} + \frac{h''(r)\cdot r - h'(r)}{r^2}\cdot r_y \cdot y$$

$$= \frac{h'(r)}{r} + \frac{h''(r)\cdot r - h'(r)}{r^2}\cdot\frac{y^2}{r} \longrightarrow 1pt$$

$$u_{xx}+u_{yy} = \frac{2h'(r)}{r} + \frac{h''(r)\cdot r - h'(r)}{r^2}\cdot\frac{x^2+y^2}{r}$$

$$= \frac{h'(r)}{r} + h''(r)$$

$x^2+y^2=r^2$

So: $h''(r) + \frac{1}{r}h'(r) = r^2$

$\longrightarrow$ 1pt.

2. $\boxed{7 \text{ pts}}$

$$\begin{cases} x_t = 1 \\ y_t = 2 \\ u_t = 3 \end{cases} \longrightarrow 1pt$$

$$\begin{cases} x(0,s) = s \\ y(0,s) = 0 \\ u(0,s) = s. \end{cases} \longrightarrow 1pt$$

$$\Rightarrow \quad x(t,s) = \int_0^t 1\,dt + x(0,s)$$
$$= t + s \longrightarrow 1pt$$

$$y(t,s) = \int_0^t 2\,dt + y(0,s)$$
$$= 2t + 0 = 2t \longrightarrow 1pt$$

$$u(t,s) = \int_0^t 3\,dt + u(0,s) = 3t + s. \longrightarrow 1pt$$

$$t = \frac{y}{2}, \quad s = x - t = x - \frac{y}{2} \longrightarrow 1pt$$

$$\text{So } u(x,y) = \frac{3y}{2} + x - \frac{y}{2} = x + y \longrightarrow 1pt$$

3.

(a) $\boxed{2\text{pts}}$ $\frac{dy}{dx} = \frac{3}{1} = 3$ $\longrightarrow$ 1pt

$y = 3x + C$ $\longrightarrow$ 1pt

(b) $\boxed{4\text{pts}}$ on $\Gamma(s)$: $\begin{cases} x = s \\ y = 3s \\ u = 1 \end{cases}$

$J = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0.$ $\longrightarrow$ 1pt

$\Rightarrow \begin{cases} \infty \text{ many solution} \\ \text{or no solution} \end{cases}$ $\longrightarrow$ 1pt

- along $\Gamma(s)$

$\longrightarrow$ 1pt

(i) $\frac{du}{ds} = u_x x_s + u_y y_s = u_x + 3u_y = u^3 = 1$

(ii) since $u=1$, $\frac{du}{ds} = 0$. contradiction

$\Rightarrow$ no solution. $\longrightarrow$ 1pt

4. $\boxed{5pts}$

$X_n = \cos(n\pi x), \quad n=0,1,2,3\cdots$

$u = \sum_{n=0}^{\infty} X_n T_n \quad \longrightarrow 1pt$

$\sum_{n=0}^{\infty} T_n'(t)\cos(n\pi x) + \sum_{n=0}^{\infty} T_n\, n^2\pi^2\cos(n\pi x)$

$= \cos t \quad \longrightarrow 1pt$

$$\begin{cases} T_0'(t) = \cos t \\ T_n'(t) + n^2\pi^2 T_n(t) = 0. \end{cases} \quad \longrightarrow 1pt$$

$T_0 = \sin(t) + A_0.$

$T_n = A_n e^{-n^2\pi^2 t} \quad \nearrow 1pt.$

$\Rightarrow u = \sin(t) + A_0 + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t}\cos(n\pi x).$

$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$

$= 1 + \cos(\pi x)$

$\Rightarrow A_0 = 1, \quad A_1 = 1, \quad A_n = 0, \ n=2,3,\cdots$

$$\Rightarrow u(x,t) = \sin(t) + 1 + e^{-\pi^2 t} \cos(\pi x).$$

$\longrightarrow$ 1pt.

$\boxed{5\text{pts}}$

5. $c=1, \quad L=1$

$$u = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \cos(n\pi x) \left[A_n \cdot \cos(n\pi t) + B_n \sin(n\pi t) \right] \rightarrow 1pt$$

$$u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

$$= 1 + \cos(\pi x) \rightarrow 1pt$$

$A_0 = 2, \quad A_1 = 1, \quad A_n = 0, \quad n = 2, 3, \ldots$

$$u_t(x,t) = \frac{B_0}{2} + \sum_{n=1}^{\infty} \cos(n\pi x) \cdot \left[-A_n \sin(n\pi t)\, n\pi + B_n \cdot n\pi \cos(n\pi t) \right]$$

$$\Rightarrow u_t(x,0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} n\pi B_n \cos(n\pi x) \rightarrow 1pt$$

$$= \cos(\pi x)\cos(2\pi x) = \frac{1}{2}\left[\cos(3\pi x) + \cos(\pi x)\right]$$

$\Rightarrow B_0 = 0, \qquad \pi B_1 = \frac{1}{2} \Rightarrow B_1 = \frac{1}{2\pi}$

$3\pi B_3 = \frac{1}{2} \Rightarrow B_3 = \frac{1}{6\pi}$

$B_n = 0, \quad \text{if } n \neq 1, 3. \quad \longrightarrow 1pt$

$$\Rightarrow u = 1 + \cos(\pi x) \cdot \left[\cos(\pi t) + \frac{1}{2\pi}\sin(\pi t)\right]$$

$$+ \frac{1}{6\pi}\cos(3\pi x) \cdot \sin(3\pi t).$$

$\longrightarrow 1pt$