MATH 146C discussion — Ryan Ta
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Solutions to suggested homework problems from
An Introduction to Partial Differential Equations by Yehuda Pinchover and Jacob Rubinstein

Suggested problems: Exercises 1.1, 1.2, 1.4, 1.7

1.1. Show that each of the following equations has a solution of the form $u(x,y) = f(ax+by)$ for a proper choice of constants a, b. Find the constants for each example.

(a) $u_x + 3u_y = 0$

Solution. Given $u(x,y) = f(ax+by)$, we obtain the first partial derivatives

$$\begin{aligned} u_x(x,y) &= \frac{\partial}{\partial x} f(ax+by) \\ &= f'(ax+by)\frac{\partial}{\partial x}(ax+by) \\ &= af'(ax+by) \end{aligned}$$

and

$$\begin{aligned} u_y(x,y) &= \frac{\partial}{\partial y} f(ax+by) \\ &= f'(ax+by)\frac{\partial}{\partial y}(ax+by) \\ &= bf'(ax+by). \end{aligned}$$

So we have

$$\begin{aligned} u_x + 3u_y &= af'(ax+by) + 3bf'(ax+by) \\ &= (a+3b)bf'(ax+by) \\ &= 0 \end{aligned}$$

provided we set $a + 3b = 0$. In other words, $u(x,y) = f(ax+by)$ is a solution of $u_x + 3u_y = 0$ if we choose the constants a, b that satisfy $a + 3b = 0$. □

(b) $3u_x - 7u_y = 0$

Solution. Given $u(x,y) = f(ax+by)$, we obtain the first partial derivatives

$$\begin{aligned} u_x(x,y) &= af'(ax+by), \\ u_y(x,y) &= bf'(ax+by). \end{aligned}$$

So we have

$$\begin{aligned} 3u_x - 7u_y &= 3af'(ax+by) - 7bf'(ax+by) \\ &= (3a-7b)f'(ax+by) \\ &= 0 \end{aligned}$$

provided we set $3a - 7b = 0$. In other words, $u(x,y) = f(ax+by)$ is a solution of $3u_x - 7u_y = 0$ if we choose the constants a, b that satisfy $3a - 7b = 0$. □

(c) $2u_x + \pi u_y = 0$

Solution. Given $u(x,y) = f(ax+by)$, we obtain the first partial derivatives

$$\begin{aligned} u_x(x,y) &= af'(ax+by), \\ u_y(x,y) &= bf'(ax+by). \end{aligned}$$

So we have

$$\begin{aligned} 2u_x + \pi u_y &= 2af'(ax+by) + \pi bf'(ax+by) \\ &= (2a+\pi b)f'(ax+by) \\ &= 0 \end{aligned}$$

provided we set $2a + \pi b = 0$. In other words, $u(x,y) = f(ax+by)$ is a solution of $2u_x + \pi u_y = 0$ if we choose the constants a, b that satisfy $2a + \pi b = 0$. □

1.2. Show that each of the following equations has a solution of the form $u(x, y) = e^{\alpha x+\beta y}$. Find the constants α, β for each example.

(a) $u_x + 3u_y + u = 0$

Solution. Given $u(x, y) = e^{\alpha x+\beta y}$, we obtain the first partial derivatives

$$\begin{aligned} u_x(x, y) &= \frac{\partial}{\partial x}(e^{\alpha x+\beta y}) \\ &= e^{\alpha x+\beta y}\frac{\partial}{\partial x}(\alpha x + \beta y) \\ &= \alpha e^{\alpha x+\beta y} \end{aligned}$$

and

$$\begin{aligned} u_y(x, y) &= \frac{\partial}{\partial y}(e^{\alpha x+\beta y}) \\ &= e^{\alpha x+\beta y}\frac{\partial}{\partial y}(\alpha x + \beta y) \\ &= \beta e^{\alpha x+\beta y}. \end{aligned}$$

So we have

$$\begin{aligned} 0 &= u_x + 3u_y + u \\ &= (\alpha e^{\alpha x+\beta y}) + 3(\beta e^{\alpha x+\beta y}) + e^{\alpha x+\beta y} \\ &= (\alpha + 3\beta + 1)e^{\alpha x+\beta y}, \end{aligned}$$

provided we set $\alpha + 3\beta + 1 = 0$. In other words, $u(x, y) = e^{\alpha x+\beta y}$ is a solution of $u_x + 3u_y + u = 0$ if we choose the constants α, β that satisfy $\alpha + 3\beta + 1 = 0$. □

(b) $u_{xx} + u_{yy} = 5e^{x-2y}$

Solution. Given $u(x, y) = e^{\alpha x+\beta y}$, we obtain the first partial derivatives

$$\begin{aligned} u_x(x, y) &= \alpha e^{\alpha x+\beta y}, \\ u_y(x, y) &= \beta e^{\alpha x+\beta y}. \end{aligned}$$

So we have

$$\begin{aligned} 5e^{x-2y} &= u_x + 3u_y + u \\ &= (\alpha e^{\alpha x+\beta y}) + 3(\beta e^{\alpha x+\beta y}) + e^{\alpha x+\beta y} \\ &= (\alpha + 3\beta + 1)e^{\alpha x+\beta y}, \end{aligned}$$

which implies $\alpha = 1$ and $\beta = -2$. In other words, $u(x, y) = e^{x-2y}$ is a solution of $u_{xx} + u_{yy} = 5e^{x-2y}$. □

(c) $u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0$

Solution. Given $u(x, y) = e^{\alpha x+\beta y}$, we obtain the first partial derivatives

$$\begin{aligned} u_{xxxx}(x, y) &= \alpha^4 e^{\alpha x+\beta y}, \\ u_{yyyy}(x, y) &= \beta^4 e^{\alpha x+\beta y}, \\ u_{xxyy}(x, y) &= \alpha^2\beta^2 e^{\alpha x+\beta y}. \end{aligned}$$

So we have

$$\begin{aligned} 0 &= u_{xxxx} + u_{yyyy} + 2u_{xxyy} \\ &= \alpha^4 e^{\alpha x+\beta y} + \beta^4 e^{\alpha x+\beta y} + 2\alpha^2\beta^2 e^{\alpha x+\beta y} \\ &= (\alpha^4 + \beta^4 + 2\alpha^2\beta^2)e^{\alpha x+\beta y} \\ &= ((\alpha^2)^2 + 2\alpha^2\beta^2 + (\beta^2)^2)e^{\alpha x+\beta y} \\ &= (\alpha^2 + \beta^2)^2 e^{\alpha x+\beta y}, \end{aligned}$$

which implies $\alpha = 0$ and $\beta = 0$. In other words, $u(x, y) = e^{0x+0y} = 1$ is a solution of $u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0$. □

1.4. Let $u(x, y) = h(\sqrt{x^2 + y^2})$ be a solution of the minimal surface equation

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0. \tag{1.40}$$

(a) Show that $h(r)$ satisfies the ordinary differential equation

$$rh'' + h'(1 + (h'(r))^2) = 0.$$

Proof. As seen already in your lecture, we take the partial derivatives with respect to x of the known equation $r^2 = x^2+y^2$ to obtain

$$\begin{aligned} 2rr_x &= \frac{\partial}{\partial x}(r^2) \\ &= \frac{\partial}{\partial x}(x^2 + y^2) \\ &= 2x + 0 \\ &= 2x, \end{aligned}$$

which implies

$$r_x = \frac{x}{r},$$

and we repeat this process with respect to y of the known equation $r^2 = x^2 + y^2$ to obtain

$$\begin{aligned} 2rr_y &= \frac{\partial}{\partial y}(r^2) \\ &= \frac{\partial}{\partial y}(x^2 + y^2) \\ &= 0 + 2y \\ &= 2y, \end{aligned}$$

which implies

$$r_y = \frac{y}{r}.$$

So our first partial derivatives are

$$\begin{aligned} u_x(x, y) &= \frac{\partial}{\partial x}(h(r)) \\ &= h'(r)r_x \\ &= h'(r)\frac{x}{r} \end{aligned}$$

and

$$\begin{aligned} u_y(x, y) &= \frac{\partial}{\partial y}(h(r)) \\ &= h'(r)r_y \\ &= h'(r)\frac{y}{r}. \end{aligned}$$

Our second partial derivatives are

$$\begin{aligned} u_{xx}(x, y) &= \frac{\partial}{\partial x}\left(h'(r)\frac{x}{r}\right) \\ &= \frac{\partial}{\partial x}(h'(r))\frac{x}{r} + h'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right) \\ &= (h''(r)r_x)\frac{x}{r} + h'(r)\left(\frac{1}{r} - \frac{x}{r^2}r_x\right) \\ &= \left(h''(r)\frac{x}{r}\right)\frac{x}{r} + h'(r)\left(\frac{1}{r} - \frac{x}{r^2}\frac{x}{r}\right) \\ &= h''(r)\frac{x^2}{r^2} + h'(r)\frac{1}{r} - h'(r)\frac{x^2}{r^3} \end{aligned}$$

and

$$\begin{aligned} u_{yy}(x, y) &= \frac{\partial}{\partial y}\left(h'(r)\frac{y}{r}\right) \\ &= \frac{\partial}{\partial y}(h'(r))\frac{y}{r} + h'(r)\frac{\partial}{\partial y}\left(\frac{y}{r}\right) \\ &= (h''(r)r_y)\frac{y}{r} + h'(r)\left(\frac{1}{r} - \frac{y}{r^2}r_y\right) \\ &= \left(h''(r)\frac{y}{r}\right)\frac{y}{r} + h'(r)\left(\frac{1}{r} - \frac{y}{r^2}\frac{y}{r}\right) \\ &= h''(r)\frac{y^2}{r^2} + h'(r)\frac{1}{r} - h'(r)\frac{y^2}{r^3}, \end{aligned}$$

and our mixed second partial derivative is

$$\begin{aligned}
u_{xy}(x,y) &= \frac{\partial}{\partial y}\left(h'(r)\frac{x}{r}\right) \\
&= \frac{\partial}{\partial y}(h'(r))\frac{x}{r} + h'(r)\frac{\partial}{\partial y}\left(\frac{x}{r}\right) \\
&= (h''(r)r_y)\frac{x}{r} + h'(r)\left(-\frac{x}{r^2}r_y\right) \\
&= \left(h''(r)\frac{y}{r}\right)\frac{x}{r} + h'(r)\left(-\frac{x}{r^2}\frac{y}{r}\right) \\
&= h''(r)\frac{xy}{r^2} - h'(r)\frac{xy}{r^3}.
\end{aligned}$$

So we have

$$\begin{aligned}
(1+u_y^2)u_{xx} &= \left(1 + \left(h'(r)\frac{y}{r}\right)^2\right)\left(h''(r)\frac{x^2}{r^2} + h'(r)\frac{1}{r} - h'(r)\frac{x^2}{r^3}\right) \\
&= \left(1 + (h'(r))^2\frac{y^2}{r^2}\right)\left(h''(r)\frac{x^2}{r^2} + h'(r)\frac{1}{r} - h'(r)\frac{x^2}{r^3}\right) \\
&= h''(r)\frac{x^2}{r^2} + h'(r)\frac{1}{r} - h'(r)\frac{x^2}{r^3} \\
&\qquad + (h'(r))^2h''(r)\frac{x^2y^2}{r^4} + (h'(r))^3\frac{y^2}{r^3} - (h'(r))^3\frac{x^2y^2}{r^5}
\end{aligned}$$

and

$$\begin{aligned}
(1+u_x^2)u_{yy} &= \left(1 + \left(h'(r)\frac{x}{r}\right)^2\right)\left(h''(r)\frac{y^2}{r^2} + h'(r)\frac{1}{r} - h'(r)\frac{y^2}{r^3}\right) \\
&= \left(1 + (h'(r))^2\frac{x^2}{r^2}\right)\left(h''(r)\frac{y^2}{r^2} + h'(r)\frac{1}{r} - h'(r)\frac{y^2}{r^3}\right) \\
&= h''(r)\frac{y^2}{r^2} + h'(r)\frac{1}{r} - h'(r)\frac{y^2}{r^3} \\
&\qquad + (h'(r))^2h''(r)\frac{x^2y^2}{r^4} + (h'(r))^3\frac{x^2}{r^3} - (h'(r))^3\frac{x^2y^2}{r^5},
\end{aligned}$$

as well as

$$\begin{aligned}
-2u_xu_yu_{xy} &= -2\left(h'(r)\frac{x}{r}\right)\left(h'(r)\frac{y}{r}\right)\left(h''(r)\frac{xy}{r^2} - h'(r)\frac{xy}{r^3}\right) \\
&= -2(h'(r))^2\frac{xy}{r^2}\left(h''(r)\frac{xy}{r^2} - h'(r)\frac{xy}{r^3}\right) \\
&= -2(h'(r))^2h''(r)\frac{x^2y^2}{r^4} + 2(h'(r))^3\frac{x^2y^2}{r^5}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
0 &= (1+u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1+u_x^2)u_{yy} \\
&= h''(r)\frac{x^2}{r^2} + h'(r)\frac{1}{r} - h'(r)\frac{x^2}{r^3} + \cancel{(h'(r))^2h''(r)\frac{x^2y^2}{r^4}} + (h'(r))^3\frac{y^2}{r^3} - \cancel{(h'(r))^3\frac{x^2y^2}{r^5}} \\
&\qquad - \cancel{2(h'(r))^2h''(r)\frac{x^2y^2}{r^4}} + \cancel{2(h'(r))^3\frac{x^2y^2}{r^5}} \\
&\qquad + h''(r)\frac{y^2}{r^2} + h'(r)\frac{1}{r} - h'(r)\frac{y^2}{r^3} + \cancel{(h'(r))^2h''(r)\frac{x^2y^2}{r^4}} + (h'(r))^3\frac{x^2}{r^3} - \cancel{(h'(r))^3\frac{x^2y^2}{r^5}} \\
&= h''(r)\frac{x^2+y^2}{r^2} + 2h'(r)\frac{1}{r} - h'(r)\frac{x^2+y^2}{r^3} + (h'(r))^3\frac{x^2+y^2}{r^3} \\
&= h''(r)\frac{r^2}{r^2} + 2h'(r)\frac{1}{r} - h'(r)\frac{r^2}{r^3} + (h'(r))^3\frac{r^2}{r^3} \\
&= h''(r) + 2h'(r)\frac{1}{r} - h'(r)\frac{1}{r} + (h'(r))^3\frac{1}{r} \\
&= h''(r) + h'(r)\frac{1}{r} + (h'(r))^3\frac{1}{r} \\
&= h''(r) + \frac{1}{r}h'(r)(1+(h'(r))^2).
\end{aligned}$$

Finally, we can multiply both sides by r to conclude

$$rh''(r) + h'(r)(1+(h'(r))^2) = 0,$$

as desired. □

(b) What is the general solution to the equation of part (a)?

Solution. Given the ordinary differential equation

$$rh''(r) + h'(r)(1 + (h'(r))^2) = 0$$

from part (a), we can let $j(r) = h'(r)$ to rewrite the equation as

$$rj'(r) + j(r)(1 + (j(r))^2) = 0,$$

which is a separable first-order differential equation and is equivalent to

$$\frac{j'(r)}{(1 + j(r))^2} = -\frac{1}{r}.$$

By integrating both sides with respect to r, writing

$$\int \frac{j'(r)}{(1 + j(r))^2} dr = \int -\frac{1}{r} dr,$$

we obtain

$$\tan^{-1}(j(r)) = -\ln(r) + C_1,$$

or equivalently

$$h'(r) = j(r) = \tan(C_1 - \ln(r)),$$

where C_1 is an arbitrary constant. So the general solution is

$$h(r) = \boxed{\int_0^r \tan(C_1 - \ln(s))ds},$$

where we have represented this expression using the differential version of the Fundamental Theorem of Calculus. □

1.7. (a) Consider the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$. Write the equation in the coordinates $s = x$, $t = x - y$.

Solution. Define the new variables $s := x$ and $t := x - y$, which implies the first partial derivatives

$$\begin{aligned} s_x &= (x)_x = 1, \\ s_y &= (x)_y = 0, \\ t_x &= (x - y)_x = 1, \\ t_y &= (x - y)_y = -1. \end{aligned}$$

Also set $v(s,t) := u(x(s,t), y(s,t)) = u(x,y)$. Then, by the multivariable chain rule, we obtain the first partial derivatives

$$\begin{aligned} u_x &= v_s s_x + u_t t_x \\ &= v_s \cdot 1 + v_t \cdot 1 \\ &= v_s + v_t \end{aligned}$$

and

$$\begin{aligned} u_y &= v_s s_y + v_t t_y \\ &= v_s \cdot 0 + v_t \cdot (-1) \\ &= -v_t, \end{aligned}$$

as well as the second partial derivatives

$$\begin{aligned} u_{xx} &= (v_s + v_t)_x \\ &= (v_s)_x + (v_t)_x \\ &= (v_{ss}s_x + v_{st}t_x) + (v_{st}s_x + v_{tt}t_x) \\ &= v_{ss} \cdot 1 + v_{st} \cdot 1 + v_{st} \cdot 1 + v_{tt} \cdot 1 \\ &= v_{ss} + 2v_{st} + v_{tt} \end{aligned}$$

and

$$\begin{aligned} u_{xy} &= (v_s + v_t)_y \\ &= (v_s)_y + (v_t)_y \\ &= (v_{ss}s_y + v_{st}t_y) + (v_{st}s_y + v_{tt}t_y) \\ &= v_{ss} \cdot 0 + v_{st} \cdot (-1) + v_{st} \cdot 0 + v_{tt} \cdot (-1) \\ &= -v_{st} - v_{tt} \end{aligned}$$

and

$$\begin{aligned} u_{yy} &= (-v_t)_y \\ &= -(v_t)_y \\ &= -(v_{st}s_y + v_{tt}t_y) \\ &= -(v_{st} \cdot 0 + v_{tt} \cdot (-1)) \\ &= v_{tt} \end{aligned}$$

So we have

$$\begin{aligned} 0 &= u_{xx} + 2u_{xy} + u_{yy} \\ &= (v_{ss} + 2v_{st} + v_{tt}) + 2(-v_{st} - v_{tt}) + v_{tt} \\ &= v_{ss} + \cancel{2v_{st}} + \cancel{v_{tt}} - \cancel{2v_{st}} - \cancel{2v_{tt}} + \cancel{v_{tt}} \\ &= v_{ss}. \end{aligned}$$

So we have transformed the partial differential equation $u_{xx} + 2u_{xy} + u_{yy} = 0$ into

$$v_{ss} = 0,$$

as desired. □

(b) Find the general solution of the equation.

Solution. Recall from part (a) the new variables $s := x$ and $t := x - y$. The partial differential equation

$$v_{ss} = 0$$

has the general solution

$$\begin{aligned} v(s,t) &= \int \left(\int v_{ss}\, ds) \right) ds \\ &= \int \left(\int 0\, ds) \right) ds \\ &= \int f(s) ds \\ &= tf(s) + g(t). \end{aligned}$$

where $f(s)$ is an arbitrary function of s and $g(t)$ is an aribtrary function of t. Therefore, we have

$$\begin{aligned} u(x,y) &= u(x(s,t), y(s,t)) \\ &= v(s,t) \\ &= tf(s) + g(t) \\ &= \boxed{(x-y)f(x-y) + g(x-y)}, \end{aligned}$$

as desired. □

(c) Consider the equation $u_{xx} - 2u_{xy} + 5u_{yy} = 0$. Write it in the coordinates $s = x + y$, $t = 2x$.

Solution. Define the new variables $s := x + y$ and $t := 2x$, which implies the first partial derivatives

$$\begin{aligned} s_x &= (x+y)_x = 1, \\ s_y &= (x+y)_y = 1, \\ t_x &= (2x)_x = 2, \\ t_y &= (2x)_y = 0. \end{aligned}$$

Also set $v(s,t) := u(x(s,t), y(s,t)) = u(x,y)$. Then, by the multivariable chain rule, we obtain the first partial derivatives

$$\begin{aligned} u_x &= v_s s_x + u_t t_x \\ &= v_s \cdot 1 + v_t \cdot 2 \\ &= v_s + 2v_t \end{aligned}$$

and

$$\begin{aligned} u_y &= v_s s_y + v_t t_y \\ &= v_s \cdot 1 + v_t \cdot 0 \\ &= v_s, \end{aligned}$$

as well as the second partial derivatives

$$\begin{aligned} u_{xx} &= (v_s + 2v_t)_x \\ &= (v_s)_x + 2(v_t)_x \\ &= (v_{ss}s_x + v_{st}t_x) + 2(v_{st}s_x + v_{tt}t_x) \\ &= v_{ss} \cdot 1 + v_{st} \cdot 2 + 2v_{st} \cdot 1 + 2v_{tt} \cdot 2 \\ &= v_{ss} + 4v_{st} + 4v_{tt} \end{aligned}$$

and

$$\begin{aligned} u_{xy} &= (v_s + 2v_t)_y \\ &= (v_s)_y + 2(v_t)_y \\ &= (v_{ss}s_y + v_{st}t_y) + 2(v_{st}s_y + v_{tt}t_y) \\ &= v_{ss} \cdot 1 + v_{st} \cdot 0 + 2v_{st} \cdot 1 + v_{tt} \cdot 0 \\ &= v_{ss} + 2v_{st} \end{aligned}$$

and

$$\begin{aligned} u_{yy} &= (v_s)_y \\ &= v_{ss}s_y + v_{st}t_y \\ &= v_{ss} \cdot 1 + v_{st} \cdot 0 \\ &= v_{ss} \end{aligned}$$

So we have

$$\begin{aligned} 0 &= u_{xx} - 2u_{xy} + 5u_{yy} \\ &= (v_{ss} + 4v_{st} + 4v_{tt}) - 2(v_{ss} + 2v_{st}) + 5v_{ss} \\ &= v_{ss} + \cancel{4v_{st}} + 4v_{tt} - 2v_{ss} - \cancel{4v_{st}} + 5v_{ss} \\ &= 4v_{ss} + 4v_{tt} \\ &= 4(v_{ss} + v_{tt}). \end{aligned}$$

So we have transformed the partial differential equation $u_{xx} - 2u_{xy} + 5u_{yy} = 0$ into

$$v_{ss} + v_{tt} = 0,$$

as desired. □