MATH 146C discussion
University of California, Riverside

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Spring 2020

Solutions to suggested homework problems from
An Introduction to Partial Differential Equations by Yehuda Pinchover and Jacob Rubinstein

Suggested problems: Exercises 2.1, 2.2, 2.3, 2.4, 2.6, 2.7, 2.11, 2.12, 2.16, 2.21

Note: Almost all steps for solving an ordinary differential equation (for example, any material from MATH 046 at UC Riverside) are omitted from my solutions for purposes of brevity.

2.1. Consider the Cauchy problem.

$$\begin{aligned} u_x + u_y &= 1, \\ u(x,0) &= f(x). \end{aligned}$$

(a) What are the projections of the characteristic curves on the (x, y) plane?

Solution. We employ the method of characteristics for first-order partial differential equations. We parameterize the following variables:

$$\begin{aligned} x &= x(t,s), \\ y &= y(t,s), \\ u &= u(x,y) = u(x(t,s), y(t,s)) = u(t,s). \end{aligned}$$

Our characteristic equations are

$$\begin{aligned} \frac{\partial x}{\partial t} &= 1, \\ \frac{\partial y}{\partial t} &= 1, \\ \frac{\partial u}{\partial t} &= 1 \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(0,s) &= s, \\ y(0,s) &= 0, \\ u(0,s) &= u(x(0,s), y(0,s)) = u(s,0) = 1. \end{aligned}$$

Now, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{\partial y}{\partial t}}{\frac{\partial x}{\partial t}} \\ &= \frac{1}{1} \\ &= 1, \end{aligned}$$

which implies

$$\boxed{y = x + C},$$

where C is a constant. □

(b) Solve the equation.

Solution. We can solve the characteristic equations and apply the initial conditions to obtain the characteristic curves

$$\begin{aligned} x(t,s) &= t + s, \\ y(t,s) &= t, \\ u(t,s) &= t + f(s). \end{aligned}$$

The first two characteristic curves $x(t,s), y(t,s)$ imply

$$\begin{aligned} s &= x - t, \\ t &= y. \end{aligned}$$

So our solution is

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= t + f(s) \\ &= t + f(x-t) \\ &= \boxed{y + f(x-y)}, \end{aligned}$$

as desired. □

2.2. Solve the Cauchy problem

$$\begin{aligned} xu_x + (x+y)u_y &= 1, \\ u(1,y) &= y. \end{aligned}$$

Is the solution defined everywhere?

Solution. Our characteristic equations are

$$\begin{aligned} \frac{\partial x}{\partial t} &= x, \\ \frac{\partial y}{\partial t} &= x + y, \\ \frac{\partial u}{\partial t} &= 1 \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(0,s) &= 1, \\ y(0,s) &= s, \\ u(0,s) &= u(x(0,s), y(0,s)) = u(1,s) = s. \end{aligned}$$

The first characteristic equation with the first initial condition

$$\begin{aligned} \frac{\partial x}{\partial t} &= x, \\ x(0,s) &= 1 \end{aligned}$$

implies the characteristic curve

$$x(t,s) = e^t.$$

Consequently, the second characteristic equation

$$\begin{aligned} \frac{\partial y}{\partial t} &= x + y \\ &= e^t + y \end{aligned}$$

is equivalent to the linear first-order differential equation

$$\frac{\partial y}{\partial t} - y = e^t,$$

which we can solve with the initial condition $y(0,s) = s$ to obtain the characteristic curve

$$y(t,s) = e^t(t+s).$$

Finally, the third characteristic equation and the third initial condition

$$\begin{aligned} \frac{\partial u}{\partial t} &= 1, \\ u(0,s) &= s \end{aligned}$$

implies the characteristic curve

$$u(t,s) = t + s.$$

The first and second characteristic curves imply

$$\begin{aligned} e^t &= x, \\ t + s &= \frac{y}{e^t}. \end{aligned}$$

So our solution is

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= t+s \\ &= \frac{y}{e^t} \\ &= \boxed{\frac{y}{x}}, \end{aligned}$$

as desired. We see from our expression of $u(x,y)$ that the solution is not defined on the line $x = 0$. □

2.3. Let $p \in \mathbb{R}$ be fixed. Consider the partial differential equation

$$xu_x + yu_y = pu.$$

for all $x, y \in \mathbb{R}$.

(a) Find the characteristic curves for the equations.

Solution. Our characteristic equations are

$$\begin{aligned} \frac{\partial x}{\partial t} &= x, \\ \frac{\partial y}{\partial t} &= y, \\ \frac{\partial u}{\partial t} &= p \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(0,s) &= s, \\ y(0,s) &= s, \\ u(0,s) &= s. \end{aligned}$$

We can solve the characteristic equations and apply the initial conditions to obtain the characteristic curves

$$\begin{aligned} x(t,s) &= se^t, \\ y(t,s) &= se^t, \\ u(t,s) &= se^{pt}, \end{aligned}$$

as desired. □

(b) Let $p = 4$. Find an explicit solution that satisfies $u = 1$ on the circle $x^2 + y^2 = 1$.

Solution. For a simple example, let

$$\begin{aligned} s &= 1, \\ s &= 0, \\ s &= 1, \end{aligned}$$

which satisfies

$$\begin{aligned} s^2 + s^2 &= 1^2 + 0^2 \\ &= 1. \end{aligned}$$

Then we have constructed the Cauchy problem

$$\begin{aligned} xu_x + yu_y &= 4u, \\ u(1,0) &= 1. \end{aligned}$$

Then the initial conditions for our characteristic equations from part (a) become

$$\begin{aligned} x(0,s) &= 1, \\ y(0,s) &= 0, \\ u(0,s) &= u(x(0,s), y(0,s)) = u(1,0) = 1, \end{aligned}$$

and our characteristic curves from part (a) become

$$\begin{aligned} x(t,s) &= e^t, \\ y(t,s) &= 0, \\ u(t,s) &= e^{4t}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= e^{4t} \\ &= (e^t)^4 \\ &= (x(t,s))^4 \\ &= \boxed{x^4}, \end{aligned}$$

which is one solution of the Cauchy problem. □

(c) Let $p = 2$. Find two solutions that satisfy $u(x,0) = x^2$ for every $x > 0$.

Solution. We have the Cauchy problem

$$\begin{aligned} xu_x + yu_y &= 2u, \\ u(x,0) &= x^2. \end{aligned}$$

Then the initial conditions for our characteristic equations from part (a) become

$$\begin{aligned} &x(0,s) = s, \\ &y(0,s) = 0, \\ &u(0,s) = u(x(0,s), y(0,s)) = u(s,0) = s^2, \end{aligned}$$

and our characteristic curves from part (a) become

$$\begin{aligned} x(t,s) &= se^t, \\ y(t,s) &= 0, \\ u(t,s) &= s^2e^{2t}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= s^2e^{2t} \\ &= (se^t)^2 \\ &= (x(t,s))^2 \\ &= \boxed{x^2}, \end{aligned}$$

which is one solution of the Cauchy problem. We also observe that

$$u(x,y) = \boxed{x^2 + y^2}$$

is another solution of the Cauchy problem because we have

$$\begin{aligned} xu_x + yu_y &= x\frac{\partial}{\partial x}(x^2+y^2) + y\frac{\partial}{\partial y}(x^2+y^2) \\ &= x(2x) + y(2y) \\ &= 2(x^2+y^2) \\ &= 2u \end{aligned}$$

and

$$\begin{aligned} u(x,0) &= x^2 + 0^2 \\ &= x^2. \end{aligned}$$

So we have found two solutions of the Cauchy problem. □

(d) Explain why the result in (c) does not contradict the existence-uniqueness theorem.

Solution. The existence-uniqueness theorem (page 36 of the textbook) basically states:

- If the transversality condition holds for all $s \in (s_0 - 2\delta, s_0 + 2\delta)$, for $\delta > 0$, then there exists a unique solution of the Cauchy problem in the neighborhood of $(t, s) \in (-\epsilon, \epsilon) \times (s_0 - \delta, s_0 + \delta)$, for $\epsilon > 0$.
- If there exists $s \in (s_0 - 2\delta, s_0 + 2\delta)$ such that the transversality condition does not hold, then the Cauchy problem has either no solution or infinitely many solutions.

Furthermore, the transversality condition of the Cauchy problem holds if we have

$$\begin{aligned} J|_{t=0} &= \begin{vmatrix} \frac{\partial}{\partial t}x(0,s) & \frac{\partial}{\partial t}y(0,s) \\ \frac{\partial}{\partial s}x(0,s) & \frac{\partial}{\partial s}y(0,s) \end{vmatrix} \\ &= \frac{\partial}{\partial t}x(0,s)\frac{\partial}{\partial s}y(0,s) - \frac{\partial}{\partial t}y(0,s)\frac{\partial}{\partial s}x(0,s) \\ &\neq 0. \end{aligned}$$

But, for this problem, the Jacobian is

$$\begin{aligned} J|_{t=0} &= \begin{vmatrix} \frac{\partial}{\partial t}x(0,s) & \frac{\partial}{\partial t}y(0,s) \\ \frac{\partial}{\partial s}x(0,s) & \frac{\partial}{\partial s}y(0,s) \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial t}x(t,s)|_{t=0} & \frac{\partial}{\partial t}y(t,s)|_{t=0} \\ \frac{\partial}{\partial s}x(t,s)|_{t=0} & \frac{\partial}{\partial s}y(t,s)|_{t=0} \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial t}(se^t)|_{t=0} & \frac{\partial}{\partial t}(se^t)|_{t=0} \\ \frac{\partial}{\partial s}(se^t)|_{t=0} & \frac{\partial}{\partial s}(se^t)|_{t=0} \end{vmatrix} \\ &= \begin{vmatrix} se^t|_{t=0} & se^t|_{t=0} \\ e^t|_{t=0} & e^t|_{t=0} \end{vmatrix} \\ &= \begin{vmatrix} se^0 & se^0 \\ e^0 & e^0 \end{vmatrix} \\ &= \begin{vmatrix} s & s \\ 1 & 1 \end{vmatrix} \\ &= s \cdot 1 - 1 \cdot s \\ &= 0, \end{aligned}$$

meaning that the transversality condition does not hold for this problem. So there does not exist only one solution of this problem. The reason that part (c) does not contradict the existence-uniqueness theorem is that we found two solutions. The existence-uniqueness theorem suggests that this Cauchy problem has in fact infinitely many solutions. □

2.4. Consider the equation

$$yu_x - xu_y = 0$$

for all $y > 0$. Check for each of the following initial conditions whether the Cauchy problem is solvable. If it is solvable, find a solution. If it is not, explain why.

(a) $u(x, 0) = x^2$.

Solution. We claim that the Cauchy problem

$$\begin{aligned} yu_x - xu_y &= 0, \\ u(x,0) &= x^2 \end{aligned}$$

has a solution. Our characteristic equations are

$$\begin{aligned} \frac{\partial x}{\partial t} &= y, \\ \frac{\partial y}{\partial t} &= -x, \\ \frac{\partial u}{\partial t} &= 0 \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(0,s) &= s, \\ y(0,s) &= 0, \\ u(0,s) &= u(x(0,s), y(0,s)) = u(s,0) = s^2. \end{aligned}$$

First, we notice

$$\begin{aligned}\frac{\partial^2 x}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t}\right)\\ &= \frac{\partial y}{\partial t}\\ &= -x,\end{aligned}$$

which is equivalent to the second-order equation

$$\frac{\partial^2 x}{\partial t^2} + x = 0,$$

from which we can solve in t to obtain

$$x(t,s) = C_1(s)\cos(t) + C_2(s)\sin(t),$$

where $C_1(s), C_2(s)$ are both constant in t. Applying the initial conditions

$$\begin{aligned}x(0,s) &= s,\\ \frac{\partial x}{\partial t}(0,s) &= y(0,s) = 0\end{aligned}$$

gives $C_1(s) = s$ and $C_2(s) = 0$, and so we get

$$x(t,s) = s\cos(t).$$

Next, we notice

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)\\ &= \frac{\partial}{\partial t}(-x)\\ &= -\frac{\partial x}{\partial t}\\ &= -y,\end{aligned}$$

which is equivalent to the second-order equation

$$\frac{\partial^2 y}{\partial t^2} + y = 0,$$

from which we can solve in t to obtain

$$y(t,s) = C_3(s)\cos(t) + C_4(s)\sin(t),$$

where $C_3(s), C_4(s)$ are both constant in t. Applying the initial conditions

$$\begin{aligned}y(0,s) &= 0,\\ \frac{\partial y}{\partial t}(0,s) &= -x(0,s) = -s\end{aligned}$$

gives $C_3(s) = 0$ and $C_4(s) = -s$, and so we get

$$y(t,s) = -s\sin(t).$$

Observe that we get

$$\begin{aligned}s^2 &= s^2(\cos^2 t + \sin^2 t)\\ &= (s\cos(t))^2 + (-s\sin(t))^2\\ &= (x(t,s))^2 + (y(t,s))^2\\ &= x^2 + y^2.\end{aligned}$$

Finally, the third characteristic equation and the third initial condition

$$\begin{aligned}\frac{\partial u}{\partial t} &= 0,\\ u(0,s) &= s^2\end{aligned}$$

implies the characteristic curve

$$u(t,s) = s^2.$$

Therefore, our solution is

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= s^2 \\ &= \boxed{x^2 + y^2}, \end{aligned}$$

as desired. □

(b) $u(x,0) = x$.

Proof. We claim that the Cauchy problem

$$\begin{aligned} yu_x - xu_y &= 0, \\ u(x,0) &= x \end{aligned}$$

does not have a solution on the entire (x,y) plane. Suppose instead that there exists a solution $u(x,y)$ to this problem. Then our characteristic equations are

$$\begin{aligned} \frac{\partial x}{\partial t} &= y, \\ \frac{\partial y}{\partial t} &= -x, \\ \frac{\partial u}{\partial t} &= 0 \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(0,s) &= s, \\ y(0,s) &= 0, \\ u(0,s) &= u(x(0,s), y(0,s)) = u(s,0) = s. \end{aligned}$$

We have already shown in part (a) that the first two characteristic curves are

$$\begin{aligned} x(t,s) &= s\cos(t), \\ y(t,s) &= -s\sin(t). \end{aligned}$$

The third characteristic equation and third initial condition

$$\begin{aligned} \frac{\partial u}{\partial t} &= 0, \\ u(0,s) &= s \end{aligned}$$

implies the curve

$$u(t,s) = s.$$

Therefore, if the solution exists, our solution would be

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= s \end{aligned}$$

for all $(x,y) \in \mathbb{R}^2$. At the same time, according to the initial condition $u(x,0) = x$, we found one point $(-s,0) \in \mathbb{R}^2$ that gives

$$u(-s,0) = -s,$$

which contradicts $u(x,y) = s$ for all $(x,y) \in \mathbb{R}^2$ if $s \neq 0$. So we conclude that the solution on the entire (x,y) plane to the Cauchy problem does not exist. □

(c) $u(x,0) = x$, where $x > 0$.

Solution. We have already shown in parts (a) and (b) that the characteristic curves are

$$\begin{aligned} x(t,s) &= s\cos(t), \\ y(t,s) &= -s\sin(t), \\ u(t,s) &= s. \end{aligned}$$

We have also already shown in part (a)

$$s^2 = x^2 + y^2,$$

which implies

$$s = \sqrt{x^2 + y^2}$$

on the domain $x > 0$. Therefore, our solution is

$$\begin{aligned} u(x, y) &= u(t, s) \\ &= s \\ &= \boxed{\sqrt{x^2 + y^2}}, \end{aligned}$$

as ddesired. □

2.6. Consider the Cauchy problem

$$\begin{aligned} xu_x + (x^2 + y)u_y + \left(\frac{y}{x} - x\right) u &= 1, \\ u(1, y) &= 0. \end{aligned}$$

(a) Solve the Cauchy problem for $x > 0$. Compute $u(3, 6)$.

Solution. Our characteristic equations are

$$\begin{aligned} \frac{\partial x}{\partial t} &= x, \\ \frac{\partial y}{\partial t} &= x^2 + y, \\ \frac{\partial u}{\partial t} &= \left(x - \frac{y}{x}\right) u + 1 \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(0, s) &= 1, \\ y(0, s) &= s, \\ u(0, s) &= u(x(0, s), y(0, s)) = u(1, s) = 0. \end{aligned}$$

The first characteristic equation

$$\frac{\partial x}{\partial t} = x$$

with the first initial condition $x(0, s) = 1$ implies the characteristic curve

$$x(t, s) = e^t.$$

The second characteristic equation

$$\begin{aligned} \frac{\partial y}{\partial t} &= x^2 + y \\ &= e^{2t} + y \end{aligned}$$

is equivalent to the first-order ordinary differential equation

$$\frac{\partial y}{\partial t} - y = e^{2t},$$

which we can solve and apply the initial condition $y(0, s) = s$ to obtain the characteristic curve

$$y(t, s) = e^{2t} + (s - 1)e^t.$$

Finally, the third characteristic equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left(x - \frac{y}{x}\right) u + 1 \\ &= \left(e^t - \frac{e^{2t} + (s - 1)e^t}{e^t}\right) u + 1 \\ &= (1 - s)u + 1 \end{aligned}$$

is equivalent to the first-order ordinary differential equation

$$\frac{\partial u}{\partial t} + (s-1)u = 1,$$

which we can solve and apply the initial condition $u(0,s) = 0$ to obtain the characteristic curve

$$u(t,s) = \frac{1}{s-1}(1 - e^{-(s-1)t}).$$

The first two characteristic curves $x(t,s), y(t,s)$ imply

$$\begin{aligned} e^t &= x, \\ s - 1 &= \frac{y - e^{2t}}{e^t} = \frac{y - x^2}{x}. \end{aligned}$$

So our solution is

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= \frac{1}{s-1}(1 - (e^t)^{-(s-1)}) \\ &= \boxed{\frac{x}{y-x^2}(1 - x^{-\frac{y-x^2}{x}})}. \end{aligned}$$

Using this solution, we also compute

$$\begin{aligned} u(3,6) &= \frac{3}{6-3^2}(1 - 3^{-\frac{6-3^2}{3}}) \\ &= \frac{3}{-3}(1 - 3^{-\frac{-3}{3}}) \\ &= -(1-3) \\ &= \boxed{2}, \end{aligned}$$

as desired. □

(b) Is the solution defined for the entire ray $x > 0$?

Answer. The "entire ray $x > 0$" means the line $\{(x,y) : \mathbb{R}^2 : x > 0, y = 0\}$. Although the solution $u(x,y)$ is not defined on the parabola $y = x^2$, the parabola does not intersect the ray. Therefore, $u(x,y)$ is defined on the entire ray $x > 0$. □

2.7. Solve the Cauchy problem

$$\begin{aligned} u_x + u_y &= u^2, \\ u(x,0) &= 1. \end{aligned}$$

Solution. Our characteristic equations are

$$\begin{aligned} \frac{\partial x}{\partial t} &= 1, \\ \frac{\partial y}{\partial t} &= 1, \\ \frac{\partial u}{\partial t} &= u^2 \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(0,s) &= s, \\ y(0,s) &= 0, \\ u(0,s) &= u(x(0,s), y(0,s)) = u(s,0) = 1. \end{aligned}$$

We can solve the characteristic equations and apply the initial conditions to obtain the characteristic curves

$$\begin{aligned} x(t,s) &= t + s, \\ y(t,s) &= t, \\ u(t,s) &= \frac{1}{1-t}. \end{aligned}$$

The first two characteristic curves $x(t,s), y(t,s)$ imply

$$\begin{aligned} s &= x - t, \\ t &= y. \end{aligned}$$

So our solution is

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= \frac{1}{1-t} \\ &= \boxed{\frac{1}{1-y}}, \end{aligned}$$

as desired. □

2.11. Solve the Cauchy problem

$$\begin{aligned} (y^2+u)u_x + yu_y &= 0, \\ u\left(\frac{y^2}{2}, y\right) &= 0 \end{aligned}$$

in the domain $y > 0$.

Solution. Our characteristic equations are

$$\begin{aligned} \frac{\partial x}{\partial t} &= y^2 + u, \\ \frac{\partial y}{\partial t} &= y, \\ \frac{\partial u}{\partial t} &= 0 \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(0,s) &= \frac{(y(0,s))^2}{2} = \frac{s^2}{2}, \\ y(0,s) &= s, \\ u(0,s) &= u(x(0,s), y(0,s)) = u\left(\frac{s^2}{2}, s^2\right) = 0. \end{aligned}$$

We can solve the second and third characteristic equations and apply the initial conditions to obtain the characteristic curves

$$\begin{aligned} y(t,s) &= se^t, \\ u(t,s) &= 0. \end{aligned}$$

So our solution is

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= \boxed{0}. \end{aligned}$$

But we claim that this is not the only solution to the problem. To see this, we will check whether the transversality condition holds for this problem. The Jacobian is

$$\begin{aligned} J|_{t=0} &= \begin{vmatrix} \frac{\partial}{\partial t}x(0,s) & \frac{\partial}{\partial t}y(0,s) \\ \frac{\partial}{\partial s}x(0,s) & \frac{\partial}{\partial s}y(0,s) \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial t}(\frac{s^2}{2}) & \frac{\partial}{\partial t}(s^2) \\ \frac{\partial}{\partial s}(\frac{s^2}{2}) & \frac{\partial}{\partial s}(s^2) \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 \\ s & 2s \end{vmatrix} \\ &= 0 \cdot (2s) - s \cdot 0 \\ &= 0, \end{aligned}$$

meaning that the transversality condition does not hold for this problem. This means that, according to the existence-uniqueness theorem, this problem has either no solution or infinitely many solutions. But we have already established that $u(x,y) = 0$ is a solution of the problem. So we conclude that this problem has infinitely many solutions. □

2.12. Solve the Cauchy problem

$$u^2u_x + u_y = 0,$$
$$u(x,0) = \sqrt{x}$$

in the ray $x > 0$. What is the domain of existence of the solution?

Solution. Our characteristic equations are

$$\frac{\partial x}{\partial t} = u^2,$$
$$\frac{\partial y}{\partial t} = 1,$$
$$\frac{\partial u}{\partial t} = 0$$

with the initial conditions

$$x(0,s) = s,$$
$$y(0,s) = 0,$$
$$u(0,s) = u(x(0,s), y(0,s)) = u(s,0) = \sqrt{s}.$$

We can solve the second and third characteristic equations and apply the initial conditions to obtain the characteristic curves

$$y(t,s) = t,$$
$$u(t,s) = \sqrt{s}.$$

So the first characteristic equation becomes

$$\begin{aligned}\frac{\partial x}{\partial t} &= u^2 \\ &= \left(\sqrt{s}\right)^2 \\ &= s,\end{aligned}$$

which we can solve and apply the initial condition $x(0,s) = s$ to obtain the first characteristic curve

$$x(t,s) = s(t+1).$$

The first two characteristic curves $x(t,s), y(t,s)$ imply

$$s = \frac{x}{t+1},$$
$$t = y.$$

So our solution is

$$\begin{aligned}u(x,y) &= u(t,s) \\ &= \sqrt{s} \\ &= \sqrt{\frac{x}{t+1}} \\ &= \boxed{\sqrt{\frac{x}{y+1}}},\end{aligned}$$

as desired. □

2.16. Solve the Cauchy problem

$$xu_x + yu_y = -u,$$
$$u(\cos(s), \sin(s)) = 1$$

for all $0 \leq s \leq \pi$. Is the solution defined everywhere?

Solution. Our characteristic equations are

$$\frac{\partial x}{\partial t} = x,$$
$$\frac{\partial y}{\partial t} = y,$$
$$\frac{\partial u}{\partial t} = -u$$

with the initial conditions

$$\begin{aligned} x(0,s) &= \cos(s), \\ y(0,s) &= \sin(s), \\ u(0,s) &= u(x(0,s), y(0,s)) = u(\cos(s), \sin(s)) = 1. \end{aligned}$$

We can solve the characteristic equations and apply the initial conditions to obtain the characteristic curves

$$\begin{aligned} x(t,s) &= e^t \cos(s), \\ y(t,s) &= e^t \sin(s), \\ u(t,s) &= e^{-t}. \end{aligned}$$

The first two characteristic curves $x(t,s), y(t,s)$ imply

$$\begin{aligned} x^2 + y^2 &= (x(t,s))^2 + (y(t,s))^2 \\ &= (e^t \cos(s))^2 + (e^t \sin(s))^2 \\ &= e^{2t}(\cos^2 s + \sin^2 s) \\ &= e^{2t}, \end{aligned}$$

which implies

$$e^t = \sqrt{x^2 + y^2}$$

because e^t is always positive. So our solution is

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= e^{-t} \\ &= \frac{1}{e^t} \\ &= \boxed{\frac{1}{\sqrt{x^2+y^2}}}. \end{aligned}$$

This solution is not defined everywhere because it is not defined on the point $(0,0)$. □

2.21. (a) Find a function $u(x,y)$ that solves the Cauchy problem

$$\begin{aligned} xu_x - yu_y &= u + xy, \\ u(x,x) &= x^2 \end{aligned}$$

for all $1 \le x \le 2$.

Solution. Our characteristic equations are

$$\frac{\partial x}{\partial t} = x,$$
$$\frac{\partial y}{\partial t} = -y,$$
$$\frac{\partial u}{\partial t} = u + xy$$

with the initial conditions

$$\begin{aligned} x(0,s) &= s, \\ y(0,s) &= x(0,s) = s, \\ u(0,s) &= u(x(0,s), y(0,s)) = u(s,s) = s^2. \end{aligned}$$

We can solve the first two characteristic equations and apply the initial conditions to obtain the characteristic curves

$$x(t,s) = se^t,$$
$$y(t,s) = se^{-t}.$$

The first two characteristic curves $x(t,s), y(t,s)$ imply

$$\begin{aligned} xy &= x(t,s)y(t,s) \\ &= (se^t)(se^{-t}) \\ &= s^2, \end{aligned}$$

which implies

$$s = \sqrt{xy}$$

if we also impose $y > 0$. (Note that $1 \le x \le 2$ clearly implies $x > 0$. So we also need $y > 0$ in order for $\sqrt{xy}$ to stay a real number.) We also obtain

$$\begin{aligned} e^t &= \frac{x}{s} \\ &= \frac{x}{\sqrt{xy}}, \end{aligned}$$

again as long as we maintain $y > 0$. Now, the third characteristic equation becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= u + xy \\ &= u + s^2, \end{aligned}$$

which is equivalent to the first-order ordinary differential equation

$$\frac{\partial u}{\partial t} - u = s^2,$$

which we can solve this equation and apply the inital condition $u(0,s) = s^2$ to obtain the third characteristic curve

$$u(t,s) = s^2(2e^t - 1).$$

So our solution is

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= s^2(2e^t - 1) \\ &= xy\left(2\frac{x}{\sqrt{xy}} - 1\right) \\ &= \boxed{2x\sqrt{xy} - xy}, \end{aligned}$$

as desired. □

(b) Check whether the transversality condition holds.

Solution. The Jacobian is

$$\begin{aligned} J|_{t=0} &= \begin{vmatrix} \frac{\partial}{\partial t}x(0,s) & \frac{\partial}{\partial t}y(0,s) \\ \frac{\partial}{\partial s}x(0,s) & \frac{\partial}{\partial s}y(0,s) \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial t}x(t,s)|_{t=0} & \frac{\partial}{\partial t}y(t,s)|_{t=0} \\ \frac{\partial}{\partial s}x(t,s)|_{t=0} & \frac{\partial}{\partial s}y(t,s)|_{t=0} \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial t}(se^t)|_{t=0} & \frac{\partial}{\partial t}(se^{-t})|_{t=0} \\ \frac{\partial}{\partial s}(se^t)|_{t=0} & \frac{\partial}{\partial s}(se^{-t})|_{t=0} \end{vmatrix} \\ &= \begin{vmatrix} se^t|_{t=0} & -se^{-t}|_{t=0} \\ e^t|_{t=0} & e^{-t}|_{t=0} \end{vmatrix} \\ &= \begin{vmatrix} se^0 & -se^{-0} \\ e^0 & e^{-0} \end{vmatrix} \\ &= \begin{vmatrix} s & -s \\ 1 & 1 \end{vmatrix} \\ &= s \cdot 1 - 1 \cdot (-s) \\ &= 2s \\ &\neq 0, \end{aligned}$$

if $s \neq 0$, meaning that the transversality condition holds for this problem if $x = x(t, s) = s \neq 0$. Namely, the transversality condition holds on $1 \leq x \leq 2$. According to the existence-uniqueness theorem, the function

$$u(x, y) = 2x\sqrt{xy} - xy$$

is the unique solution of this problem on $1 \leq x \leq 2$. □

(c) Draw the projections on the (x, y) plane of the initial curve and the characteristic curves emanating from the points $(1, 1, 1)$ and $(2, 2, 4)$.

Solution. First, we need to compute a family of projections on the (x, y) plane of the characteristic curves of $u(x, y)$. We follow exactly the same procedure of Exercise 2.1, part (a). We recall from part (a) of this exercise that our characteristic equations are

$$\frac{\partial x}{\partial t} = x,$$
$$\frac{\partial y}{\partial t} = -y,$$
$$\frac{\partial u}{\partial t} = u + xy.$$

Now, we have

$$\frac{dy}{dx} = \frac{\frac{\partial y}{\partial t}}{\frac{\partial x}{\partial t}} = \frac{-y}{x} = -\frac{y}{x},$$

which implies

$$y = \frac{C}{x}.$$

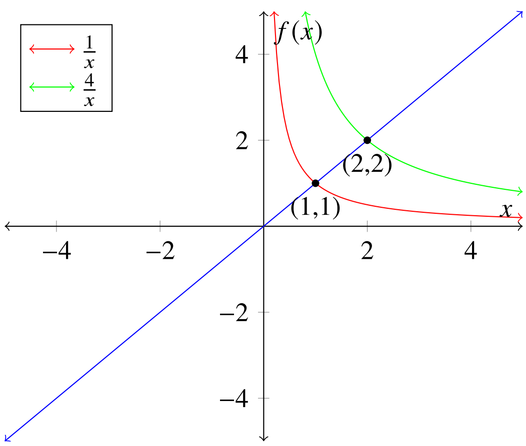
where C is a constant. The point $(1, 1, 1)$ implies the initial condition $y(1) = 1$, and so

$$y = \frac{1}{x}$$

is the projection of the characteristic curve that emanates from $(1, 1, 1)$. Similarly, the point $(2, 2, 4)$ implies the initial condition $y(2) = 2$, and so

$$y = \frac{4}{x}$$

is the projection of the characteristic curve that emanates from $(2, 2, 4)$. Now, we will draw the projections on the (x, y) plane of the initial curve and the characteristic curves emanating from the points $(1, 1, 1)$ and $(2, 2, 4)$.



The blue graph above depicts the projections on the (x, y) plane of the initial curve of $u(x, y)$. The red and green graphs above depict the projections on the (x, y) plane of the characteristic curves of $u(x, y)$ that emanate from the points $(1, 1, 1)$ and $(2, 2, 4)$, respectively. □

(d) Is the solution you found in (a) well-defined in the entire plane?

Answer. No, the solution we found in (a) is not well-defined in the entire plane. On Quadrant II ($x < 0$ and $y > 0$) and Quadrant IV ($x > 0$ and $y < 0$) of the (x, y) plane, the quantity $\sqrt{xy}$ is a purely imaginary number. □