MATH 146C discussion
University of California, Riverside

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Spring 2020

Solutions to suggested homework problems from
An Introduction to Partial Differential Equations by Yehuda Pinchover and Jacob Rubinstein

Suggested problems: Exercises 5.1, 5.3(a), 5.4(a), 5.5, 5.6, 5.7, 5.8, 5.9, 5.10(a), 5.15

Note: Almost all steps for solving an ordinary differential equation (for example, any material from MATH 046 at UC Riverside) are omitted from my solutions for purposes of brevity.

5.1. Using the method of separation of variables, find a formal solution of the problem

$$\begin{aligned} u_t &= 17u_{xx}, \\ u(0,t) &= u(\pi,t) = 0, \\ u(x,0) &= \begin{cases} 0 & \text{if } 0 \le x \le \frac{\pi}{2}, \\ 2 & \text{if } \frac{\pi}{2} < x \le \pi. \end{cases} \end{aligned}$$

for all $0 < x < \pi$ and $t > 0$.

Solution. We employ the method of separation of variables for homogeneous partial differential equations. We want to find a solution of the form

$$u(x,t) = X(x)T(t).$$

Our partial derivatives are

$$\begin{aligned} u_t(x,t) &= X(x)T_t(t), \\ u_{xx}(x,t) &= X_{xx}(x)T(t). \end{aligned}$$

So the partial differential equation

$$u_t = 17u_{xx}$$

becomes

$$X(x)T_t(t) = 17X_{xx}(x)T(t),$$

which we can algebraically rearrange to write

$$\frac{X_{xx}(x)}{X(x)} = \frac{T_t(t)}{17T(t)} = -\lambda,$$

where λ is a constant in both x and t. This produces the system of two ordinary differential equations

$$\begin{aligned} \frac{d^2X}{dx^2} + \lambda X &= 0, \\ \frac{dT}{dt} + 17\lambda T &= 0. \end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$\begin{aligned} X(x) &= \begin{cases} C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0, \\ C_1 x + C_2 & \text{if } \lambda = 0, \\ C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) & \text{if } \lambda > 0, \end{cases} \\ T(t) &= De^{-17\lambda t} \end{aligned}$$

where C_1, C_2, D are constants. Now, the boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

are equivalent to

$$\begin{aligned} X(0)T(t) &= 0, \\ X(\pi)T(t) &= 0, \end{aligned}$$

which imply either $T(t) = 0$ or $X(0) = X(\pi) = 0$. If $T(t) = 0$, then we would have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= X(x)0 \\ &= 0, \end{aligned}$$

which would be a trivial solution. But we are really interested in finding a nontrivial solution. So we should assume

$$X(0) = X(\pi) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then

$$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda} x} + C_2 e^{-\sqrt{-\lambda} x}, \\ X(0) &= 0 \end{aligned}$$

implies $C_2 = -C_1$, and so we have

$$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda} x} + C_2 e^{-\sqrt{-\lambda} x} \\ &= C_1 e^{\sqrt{-\lambda} x} - C_1 e^{-\sqrt{-\lambda} x} \\ &= C_1 (e^{\sqrt{-\lambda} x} - e^{-\sqrt{-\lambda} x}). \end{aligned}$$

Now, if $\lambda < 0$, then $e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi} \neq 0$. This means

$$\begin{aligned} X(x) &= C_1 (e^{\sqrt{-\lambda} x} - e^{-\sqrt{-\lambda} x}), \\ X(\pi) &= 0 \end{aligned}$$

implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 (e^{\sqrt{-\lambda} x} - e^{-\sqrt{-\lambda} x}) \\ &= 0(e^{\sqrt{-\lambda} x} - e^{-\sqrt{-\lambda} x}) \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= 0T(t) \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 2: Suppose $\lambda = 0$. Then

$$\begin{aligned} X(x) &= C_1 x + C_2, \\ X(0) &= 0 \end{aligned}$$

implies $C_2 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 x + C_2 \\ &= C_1 x + 0 \\ &= C_1 x. \end{aligned}$$

Next,

$$\begin{aligned} X(x) &= C_1 x, \\ X(\pi) &= 0 \end{aligned}$$

implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 x \\ &= 0 \cdot x \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= 0T(t) \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 3: Suppose $\lambda > 0$. Then

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x), \\ X(0) &= 0 \end{aligned}$$

implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \\ &= 0 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \\ &= C_2 \sin(\sqrt{\lambda}x). \end{aligned}$$

Next,

$$\begin{aligned} X(x) &= C_2 \sin(\sqrt{\lambda}x), \\ X(\pi) &= 0 \end{aligned}$$

implies $\sin(\sqrt{\lambda}\pi) = 0$, which in turn implies $\sqrt{\lambda}\pi = n\pi$, or equivalently

$$\lambda_n = \lambda = n^2,$$

and so we have

$$\begin{aligned} X_n(x) &= C_{2,n} \sin(\sqrt{\lambda_n}x) \\ &= C_{2,n} \sin(\sqrt{n^2}x) \\ &= C_{2,n} \sin(nx) \end{aligned}$$

and

$$\begin{aligned} T_n(t) &= D_n e^{-17\lambda_n t} \\ &= D_n e^{-17n^2 t} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Therefore, if we write $B_n = C_{2,n} D_n$, then we have

$$\begin{aligned} u_n(x,t) &= X_n(x) T_n(t) \\ &= (C_{2,n} \sin(nx))(D_n e^{-17n^2 t}) \\ &= C_{2,n} D_n e^{-17n^2 t} \sin(nx) \\ &= B_n e^{-17n^2 t} \sin(nx), \end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial solution, as desired.

We recall that an addition of solutions is again a solution. So that means, as we have established already that each $u_n(x,t)$ is a nontrivial solution for $n = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\ &= \sum_{n=1}^{\infty} B_n e^{-17n^2 t} \sin(nx) \end{aligned}$$

is also a solution of the problem. Finally, we need to explicitly find B_n that satisfies the initial condition

$$f(x) = u(x,0) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{\pi}{2}, \\ 2 & \text{if } \frac{\pi}{2} < x \le \pi. \end{cases}$$

Indeed, we have the Fourier coefficient

$$\begin{aligned} B_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx \\ &= \frac{2}{\pi} \left(\int_0^{\frac{\pi}{2}} 0 \sin(nx)\, dx + \int_{\frac{\pi}{2}}^{\pi} 2 \sin(nx)\, dx \right) \\ &= \frac{2}{\pi} \left(0 + \int_{\frac{\pi}{2}}^{\pi} 2 \sin(nx)\, dx \right) \\ &= \frac{4}{\pi} \int_{\frac{\pi}{2}}^{\pi} \sin(nx)\, dx \\ &= \frac{4}{\pi} \frac{1}{n} \left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) \right) \\ &= \frac{4}{\pi} \frac{1}{n} \left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n \right). \end{aligned}$$

Therefore, our solution is

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} B_n e^{-17n^2 t} \sin(nx) \\
&= \sum_{n=1}^{\infty} \left(\frac{4}{\pi} \frac{1}{n} \left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n \right) \right) e^{-17n^2 t} \sin(nx) \\
&= \boxed{\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n \right) e^{-17n^2 t} \sin(nx)},
\end{aligned}$$

as desired. □

5.3. (a) Using the method of separation of variables, find a formal solution of a vibrating string with fixed ends:

$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= 0 && \text{if } 0 < x < L, t > 0, \\
u(0,t) = u(L,t) &= 0 && \text{if } t \geq 0, \\
u(x,0) &= f(x) && \text{if } 0 \leq x \leq L, \\
u_t(x,0) &= g(x) && \text{if } 0 \leq x \leq L.
\end{aligned}$$

Solution. We want to find a solution of the form

$$u(x,t) = X(x)T(t).$$

Our partial derivatives are

$$\begin{aligned}
u_{tt}(x,t) &= X(x)T_{tt}(t), \\
u_{xx}(x,t) &= X_{xx}(x)T(t).
\end{aligned}$$

So the partial differential equation

$$u_{tt} - c^2 u_{xx} = 0$$

becomes

$$X(x)T_{tt}(t) - c^2 X_{xx}(x)T(t) = 0,$$

which we can algebraically rearrange to write

$$\frac{X_{xx}(x)}{X(x)} = \frac{T_{tt}(t)}{c^2 T(t)} = -\lambda,$$

where λ is a constant in both x and t. This produces the system of two ordinary differential equations

$$\begin{aligned}
\frac{d^2X}{dx^2} + \lambda X &= 0 \\
\frac{d^2T}{dt^2} + \lambda c^2 T &= 0.
\end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$X(x) = \begin{cases} C_1 e^{\sqrt{-\lambda} x} + C_2 e^{-\sqrt{-\lambda} x} & \text{if } \lambda < 0, \\ C_1 x + C_2 & \text{if } \lambda = 0, \\ C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x) & \text{if } \lambda > 0, \end{cases}$$

$$T(t) = \begin{cases} D_1 e^{\sqrt{-\lambda c^2} t} + D_2 e^{-\sqrt{-\lambda c^2} t} & \text{if } \lambda < 0, \\ D_1 t + D_2 & \text{if } \lambda = 0, \\ D_1 \cos(\sqrt{\lambda c^2} t) + D_2 \sin(\sqrt{\lambda c^2} t) & \text{if } \lambda > 0, \end{cases}$$

where C_1, C_2, D_1, D_2 are constants. Now, the boundary conditions

$$u(0,t) = u(L,t) = 0$$

are equivalent to

$$\begin{aligned}
X(0)T(t) &= 0, \\
X(L)T(t) &= 0,
\end{aligned}$$

which imply either $T(t) = 0$ or $X(0) = X(L) = 0$. If $T(t) = 0$, then we would have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= X(x)0 \\ &= 0, \end{aligned}$$

which would be a trivial solution. But we are really interested in finding a nontrivial solution. So we should assume

$$X(0) = X(L) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then

$$\begin{aligned} &X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}, \\ &X(0) = 0 \end{aligned}$$

  implies $C_2 = -C_1$, and so we have

$$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \\ &= C_1 e^{\sqrt{-\lambda}x} - C_1 e^{-\sqrt{-\lambda}x} \\ &= C_1 (e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}). \end{aligned}$$

  Now, if $\lambda < 0$, then $e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L} \neq 0$. This means

$$\begin{aligned} &X(x) = C_1 (e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}), \\ &X(L) = 0 \end{aligned}$$

  implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 (e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}) \\ &= 0(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}) \\ &= 0. \end{aligned}$$

  Therefore, we have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= 0T(t) \\ &= 0, \end{aligned}$$

  which is a trivial solution.

- Case 2: Suppose $\lambda = 0$. Then

$$\begin{aligned} &X(x) = C_1 x + C_2, \\ &X(0) = 0 \end{aligned}$$

  implies $C_2 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 x + C_2 \\ &= C_1 x + 0 \\ &= C_1 x. \end{aligned}$$

  Next,

$$\begin{aligned} &X(x) = C_1 x, \\ &X(L) = 0 \end{aligned}$$

  implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 x \\ &= 0 \cdot x \\ &= 0. \end{aligned}$$

  Therefore, we have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= 0T(t) \\ &= 0, \end{aligned}$$

  which is a trivial solution.

- Case 3: Suppose $\lambda > 0$. Then

$$\begin{aligned}X(x) &= C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x),\\ X(0) &= 0\end{aligned}$$

implies $C_1 = 0$, and so we have

$$\begin{aligned}X(x) &= C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)\\ &= 0\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)\\ &= C_2\sin(\sqrt{\lambda}x).\end{aligned}$$

Next,

$$\begin{aligned}X(x) &= C_2\sin(\sqrt{\lambda}x),\\ X(\pi) &= 0\end{aligned}$$

implies $\sin(\sqrt{\lambda}L) = 0$, which in turn implies $\sqrt{\lambda}L = n\pi$, or equivalently

$$\lambda_n = \lambda = \left(\frac{n\pi}{L}\right)^2,$$

and so we have

$$\begin{aligned}X_n(x) &= C_{2,n}\sin(\sqrt{\lambda_n}x)\\ &= C_{2,n}\sin\left(\frac{n\pi}{L}x\right)\end{aligned}$$

and

$$\begin{aligned}T_n(t) &= D_{1,n}\cos(\sqrt{\lambda_n c^2}t) + D_{2,n}\sin(\sqrt{\lambda_n c^2}t)\\ &= D_{1,n}\cos\left(\sqrt{\left(\frac{n\pi}{L}\right)^2 c^2}t\right) + D_{2,n}\sin\left(\sqrt{\left(\frac{n\pi}{L}\right)^2 c^2}t\right)\\ &= D_{1,n}\cos\left(\frac{cn\pi}{L}t\right) + D_{2,n}\sin\left(\frac{cn\pi}{L}t\right)\end{aligned}$$

for $n = 1, 2, 3, \ldots$. Therefore, if we write $A_n = C_{2,n}D_{1,n}$ and $B_n = C_{2,n}D_{2,n}$, then we have

$$\begin{aligned}u_n(x,t) &= X_n(x)T_n(t)\\ &= \left(C_{2,n}\sin\left(\frac{n\pi}{L}x\right)\right)\left(D_{1,n}\cos\left(\frac{cn\pi}{L}t\right) + D_{2,n}\sin\left(\frac{cn\pi}{L}t\right)\right)\\ &= \sin\left(\frac{n\pi}{L}x\right)\left(C_{2,n}D_{1,n}\cos\left(\frac{cn\pi}{L}t\right) + C_{2,n}D_{2,n}\sin\left(\frac{cn\pi}{L}t\right)\right)\\ &= \sin\left(\frac{n\pi}{L}x\right)\left(A_n\cos\left(\frac{cn\pi}{L}t\right) + B_n\sin\left(\frac{cn\pi}{L}t\right)\right)\end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial solution, as desired.

We recall that an addition of solutions is again a solution. So that means, as we have established already that each $u_n(x,t)$ is a nontrivial solution for $n = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t)\\ &= \boxed{\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left(A_n\cos\left(\frac{cn\pi}{L}t\right) + B_n\sin\left(\frac{cn\pi}{L}t\right)\right)},\end{aligned}$$

where

$$\begin{aligned}A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\,dx,\\ B_n &= \frac{2}{L}\int_0^L g(x)\sin\left(\frac{cn\pi}{L}x\right)\,dx\end{aligned}$$

for $n = 1, 2, 3, \ldots$ are the Fourier coefficients, is also a solution of the problem. □

5.4. (a) Find a formal solution of the problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{if } 0 < x < \pi, t > 0, \\ u(0,t) = u(\pi,t) &= 0 && \text{if } t \geq 0, \\ u(x,0) &= \sin^3(x) && \text{if } 0 \leq x \leq \pi, \\ u_t(x,0) &= \sin(2x) && \text{if } 0 \leq x \leq \pi. \end{aligned}$$

Solution. This problem is exactly the same as that of Exercise 5.2 with

$$\begin{aligned} c &= 1, \\ L &= \pi, \\ f(x) &= \sin^3(x), \\ g(x) &= \sin(2x). \end{aligned}$$

So we can take the $u(x,t)$ from our solution to Exercise 5.2 and substitute $c = 1$ and $L = \pi$ into it to write

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(A_n \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi}{L}t\right)\right) \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{\pi}x\right)\left(A_n \cos\left(\frac{(1)n\pi}{\pi}t\right) + B_n \sin\left(\frac{(1)n\pi}{\pi}t\right)\right) \\ &= \sum_{n=1}^{\infty} \sin(nx)(A_n \cos(nt) + B_n \sin(nt)). \end{aligned}$$

And its partial derivative with respect to t is

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} \sin(nx)(A_n \cos(nt) + B_n \sin(nt))\right) \\ &= \sum_{n=1}^{\infty} \sin(nx)\left(A_n \frac{\partial}{\partial t}\cos(nt) + B_n \frac{\partial}{\partial t}\sin(nt)\right) \\ &= \sum_{n=1}^{\infty} \sin(nx)(A_n(-n\sin(nt)) + B_n(n\cos(nt))) \\ &= \sum_{n=1}^{\infty} n\sin(nx)(-A_n \sin(nt) + B_n \cos(nt)). \end{aligned}$$

Now, we can use the given initial conditions to write

$$\begin{aligned} u(x,0) &= \sin^3(x) \\ &= \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x), \end{aligned}$$

where in the last step above we have employed the triple-angle trigonometric identity $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$, and

$$u_t(x,0) = \sin(2x).$$

Also, at $t = 0$, our solution becomes

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} \sin(nx)(A_n \cos(n(0)) + B_n \sin(n(0))) \\ &= \sum_{n=1}^{\infty} \sin(nx)(A_n \cdot 1 + B_n \cdot 0) \\ &= \sum_{n=1}^{\infty} A_n \sin(nx) \\ &= A_1 \sin(x) + A_2 \sin(2x) + A_3 \sin(3x) + \sum_{n=4}^{\infty} A_n \sin(nx), \end{aligned}$$

and the partial derivative of our solution becomes

$$\begin{aligned}
u_t(x,0) &= \sum_{n=1}^{\infty} n\sin(nx)(-A_n\sin(n(0)) + B_n\cos(n(0))) \\
&= \sum_{n=1}^{\infty} n\sin(nx)(-A_n\cdot 0 + B_n\cdot 1) \\
&= \sum_{n=1}^{\infty} nB_n\sin(nx) \\
&= 1B_1\sin(x) + 2B_2\sin(2x) + \sum_{n=3}^{\infty} nB_n\sin(nx).
\end{aligned}$$

Both our expressions of $u(x,0)$ yield

$$A_1\sin(x) + A_2\sin(2x) + A_3\sin(3x) + \sum_{n=4}^{\infty} A_n\sin(nx) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x).$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to obtain the Fourier coefficients

$$\begin{aligned}
A_1 &= \frac{3}{4}, \\
A_3 &= -\frac{1}{4}, \\
A_n &= 0
\end{aligned}$$

for $n = 2$ and for $n = 4, 5, 6, \ldots$. Similarly, both our expressions of $u_t(x,0)$ yield

$$1B_1\sin(x) + 2B_2\sin(2x) + \sum_{n=3}^{\infty} nB_n\sin(nx) = \sin(2x).$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to obtain the Fourier coefficients

$$\begin{aligned}
B_2 &= \frac{1}{2}, \\
B_n &= 0
\end{aligned}$$

for $n = 1$ and for $n = 3, 4, 5, \ldots$. Therefore, our formal solution is

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \sin(nx)(A_n\cos(nt) + B_n\sin(nt)) \\
&= \sum_{n=1}^{\infty} A_n\sin(nx)\cos(nt) + \sum_{n=1}^{\infty} B_n\sin(nx)\sin(nt) \\
&= \left(\frac{3}{4}\sin(1x)\cos(1t) + 0\sin(2x)\cos(2t) - \frac{1}{4}\sin(3x)\cos(3t) + \sum_{n=4}^{\infty} 0\sin(nx)\cos(nt)\right) \\
&\quad + \left(0\sin(1x)\sin(1t) + \frac{1}{2}\sin(2x)\sin(2t) + \sum_{n=3}^{\infty} 0\sin(nt)\right) \\
&= \boxed{\frac{3}{4}\sin(x)\cos(t) + \frac{1}{2}\sin(2x)\sin(2t) - \frac{1}{4}\sin(3x)\cos(3t)},
\end{aligned}$$

as desired. □

5.5. (a) Using the method of separation of variables, find a formal solution of the problem

$$\begin{aligned}
u_t - ku_{xx} &= 0 && \text{if } 0 < x < L, t > 0, \\
u_x(0,t) = u_x(L,t) &= 0 && \text{if } t \geq 0, \\
u(x,0) &= f(x) && \text{if } 0 \leq x \leq L.
\end{aligned}$$

Solution. We want to find a solution of the form

$$u(x,t) = X(x)T(t).$$

Our partial derivatives are

$$\begin{aligned} u_t(x,t) &= X(x)T_t(t), \\ u_{xx}(x,t) &= X_{xx}(x)T(t). \end{aligned}$$

So the partial differential equation

$$u_t - ku_{xx} = 0$$

becomes

$$X(x)T_t(t) - kX_{xx}(x)T(t) = 0,$$

which we can algebraically rearrange to write

$$\frac{X_{xx}(x)}{X(x)} = \frac{T_t(t)}{kT(t)} = -\lambda,$$

where λ is a constant in both x and t. This produces the system of two ordinary differential equations

$$\begin{aligned} \frac{d^2X}{dx^2} + \lambda X = 0 \\ \frac{dT}{dt} + \lambda kT = 0. \end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$\begin{aligned} X(x) &= \begin{cases} C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0, \\ C_1 x + C_2 & \text{if } \lambda = 0, \\ C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) & \text{if } \lambda > 0, \end{cases} \\ T(t) &= De^{-\lambda kt}, \end{aligned}$$

where C_1, C_2, D are constants. Now, the boundary conditions

$$u_x(0,t) = u_x(L,t) = 0$$

are equivalent to

$$\begin{aligned} \frac{d}{dx}X(0)T(t) = 0, \\ \frac{d}{dx}X(L)T(t) = 0, \end{aligned}$$

which imply either $T(t) = 0$ or $\frac{d}{dx}X(0) = \frac{d}{dx}X(L) = 0$. If $T(t) = 0$, then we would have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= X(x)0 \\ &= 0, \end{aligned}$$

which would be a trivial solution. But we are really interested in finding a nontrivial solution. So we should assume

$$\frac{d}{dx}X(0) = \frac{d}{dx}X(L) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then we have

  $$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}, \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

  which implies

  $$\begin{aligned} \frac{d}{dx}X(x) &= \sqrt{-\lambda}(C_1 e^{\sqrt{-\lambda}x} - C_2 e^{-\sqrt{-\lambda}x}), \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

which implies $\sqrt{-\lambda}(C_1 - C_2) = 0$. As we assumed $\lambda < 0$ in this case, we have $\sqrt{-\lambda} \neq 0$, and so we conclude $C_1 - C_2 = 0$, or equivalently $C_1 = C_2$. So we have

$$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \\ &= C_1 e^{\sqrt{-\lambda}x} + C_1 e^{-\sqrt{-\lambda}x} \\ &= C_1 (e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}). \end{aligned}$$

We notice $e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L} \neq 0$. This means

$$\begin{aligned} X(x) &= C_1 (e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}), \\ X(L) &= 0 \end{aligned}$$

implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 (e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}) \\ &= 0(e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}) \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= 0T(t) \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 2: Suppose $\lambda = 0$. Then we have

$$\begin{aligned} X(x) &= C_1 x + C_2, \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dx}X(x) &= C_1, \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

which implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 x + C_2 \\ &= 0x + C_2 \\ &= C_2, \end{aligned}$$

which already satisfies $\frac{d}{dx}X(L) = 0$. Therefore, if we write $\frac{A_0}{2} = C_2 D$, then we have

$$\begin{aligned} u_0(x,t) &= X(x)T(t) \\ &= C_2 D e^{-\lambda k(0)} \\ &= C_2 D \\ &= \frac{A_0}{2}, \end{aligned}$$

which is a nontrivial solution.

- Case 3: Suppose $\lambda > 0$. Then we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x), \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dx}X(x) &= \sqrt{\lambda}(-C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x)), \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

which implies $\sqrt{\lambda}C_2 = 0$, which in turn implies either $\sqrt{\lambda} = 0$ or $C_2 = 0$. But $\sqrt{\lambda} = 0$ implies $\lambda = 0$, which contradicts our assumption $\lambda > 0$ for this case. So we must have $C_2 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \\ &= C_1 \cos(\sqrt{\lambda}x) + 0 \sin(\sqrt{\lambda}x) \\ &= C_1 \cos(\sqrt{\lambda}x) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dx}X(x) &= \sqrt{\lambda}(-C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x)) \\ &= \sqrt{\lambda}(-C_1 \sin(\sqrt{\lambda}x) + 0 \cos(\sqrt{\lambda}x)) \\ &= -\sqrt{\lambda}C_1 \sin(\sqrt{\lambda}x), \end{aligned}$$

Next,

$$\begin{aligned} &\frac{d}{dx}X(x) = -\sqrt{\lambda}C_1 \sin(\sqrt{\lambda}x), \\ &\frac{d}{dx}X(L) = 0 \end{aligned}$$

implies $\sqrt{\lambda}C_1 \sin(\sqrt{\lambda}L) = 0$. As we assumed $\lambda > 0$ in this case, we have $\sqrt{\lambda} \neq 0$, and so we conclude $C_1 \sin(\sqrt{\lambda}L) = 0$. If $C_1 = 0$, then we would have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \\ &= 0 \cos(\sqrt{\lambda}x) + 0 \sin(\sqrt{\lambda}x) \\ &= 0, \end{aligned}$$

which would imply

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= 0T(t) \\ &= 0, \end{aligned}$$

meaning that $u(x,t)$ is a trivial solution. So we should assume $\sin(\sqrt{\lambda}L) = 0$, which implies $\sqrt{\lambda}L = n\pi$, or equivalently

$$\lambda_n = \lambda = \left(\frac{n\pi}{L}\right)^2,$$

and so we have

$$\begin{aligned} X_n(x) &= C_{1,n} \cos(\sqrt{\lambda_n}x) \\ &= C_{1,n} \cos\left(\frac{n\pi}{L}x\right) \end{aligned}$$

and

$$\begin{aligned} T_n(t) &= D_n e^{-\lambda_n kt} \\ &= D_n e^{-(\frac{n\pi}{L})^2 kt} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Therefore, if we write $A_n = C_{1,n}D_n$, then we have

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ &= \left(C_{1,n} \cos\left(\frac{n\pi}{L}x\right)\right)(D_n e^{-(\frac{n\pi}{L})^2 kt}) \\ &= C_{1,n}D_n e^{-(\frac{n\pi}{L})^2 kt} \cos\left(\frac{n\pi}{L}x\right) \\ &= A_n e^{-(\frac{n\pi}{L})^2 kt} \cos\left(\frac{n\pi}{L}x\right) \end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial solution, as desired.

We recall that an addition of solutions is again a solution. So that means, as we have established already that each $u_n(x,t)$ is a nontrivial solution for $n = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned} u(x,t) &= u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) \\ &= \boxed{\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 kt} \cos\left(\frac{n\pi}{L}x\right)}, \end{aligned}$$

where

$$A_0 = \frac{2}{L}\int_0^L f(x)\,dx,$$
$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\,dx$$

for $n = 1, 2, 3, \ldots$ are the Fourier coefficients, is also a solution of the problem. $\square$

(b) Solve the problem

$$\begin{aligned} u_t - 12u_{xx} &= 0 && \text{if } 0 < x < \pi, t > 0, \\ u_x(0,t) = u_x(\pi,t) &= 0 && \text{if } t \geq 0, \\ u(x,0) &= 1 + \sin^3(x) && \text{if } 0 \leq x \leq L. \end{aligned}$$

Solution. This problem is exactly the same as that of part (a) of this exercise with

$$\begin{aligned} k &= 12, \\ L &= \pi, \\ f(x) &= 1 + \sin^3(x). \end{aligned}$$

So we can take the $u(x,t)$ from our solution to part (a) of this exercise and substitute $k = 12$ and $L = \pi$ into it to write

$$\begin{aligned} u(x,t) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 kt}\cos\left(\frac{n\pi}{L}x\right) \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{\pi})^2(12)t}\cos\left(\frac{n\pi}{\pi}x\right) \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-12n^2t}\cos(nx). \end{aligned}$$

In order for $u(x,t)$ to satisfy $u(x,0) = 1 + \sin^3(x)$, we need to compute

$$A_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx,$$
$$A_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx$$

with $f(x) = 1 + \sin^3(x)$. Employing the trigonometric identities

$$\sin(3\theta) = 3\cos(\theta) - 4\sin^3(\theta),$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta)),$$

we have

$$\begin{aligned} \int_0^\pi 1 + \sin^3(x)\,dx &= \int_0^\pi 1 + \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)\,dx \\ &= \left(x - \frac{3}{4}\cos(x) + \frac{1}{12}\cos(3x)\right)\Bigg|_0^\pi \\ &= \left(\pi + \frac{2}{3}\right) - \left(-\frac{2}{3}\right) \\ &= \pi + \frac{4}{3} \end{aligned}$$

and

$$
\begin{aligned}
\int_0^{\pi} (1+\sin^3(x))\cos(nx)\,dx &= \int_0^{\pi} \left(1+\frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x)\right)\cos(nx)\,dx \\
&= \int_0^{\pi} \cos(nx)\,dx + \frac{3}{4}\int_0^{\pi} \sin(x)\cos(nx)\,dx - \frac{1}{4}\int_0^{\pi} \sin(3x)\cos(nx)\,dx \\
&= \int_0^{\pi} \cos(nx)\,dx + \frac{3}{8}\int_0^{\pi} \sin(x+nx)+\sin(x-nx)\,dx \\
&\quad - \frac{1}{8}\int_0^{\pi} \sin(3x+nx)+\sin(3x-nx)\,dx \\
&= \int_0^{\pi} \cos(nx)\,dx + \frac{3}{8}\int_0^{\pi} \sin((1+n)x)+\sin((1-n)x)\,dx \\
&\quad - \frac{1}{8}\int_0^{\pi} \sin((3+n)x)+\sin((3-n)x)\,dx \\
&= \left(\frac{1}{n}\sin(nx)\right)\Big|_0^{\pi} + \frac{3}{8}\left(-\frac{1}{1+n}\cos((1+n)x)-\frac{1}{1-n}\cos((1-n)x)\right)\Big|_0^{\pi} \\
&\quad - \frac{1}{8}\left(-\frac{1}{3+n}\cos((3+n)x)-\frac{1}{3-n}\cos((3-n)x)\right)\Big|_0^{\pi} \\
&= 0+\frac{3}{8}\left(-\frac{(-1)^{1+n}-1}{1+n}-\frac{(-1)^{1-n}-1}{1-n}\right)-\frac{1}{8}\left(-\frac{(-1)^{3+n}-1}{3+n}-\frac{(-1)^{3-n}-1}{3-n}\right) \\
&= \frac{3}{8}\left(\frac{1-(-1)^{1+n}}{1+n}+\frac{1-(-1)^{1-n}}{1-n}\right)-\frac{1}{8}\left(\frac{1-(-1)^{3+n}}{3+n}+\frac{1-(-1)^{3-n}}{3-n}\right) \\
&= \frac{3}{8}\begin{cases} 0 & \text{if } n=1,3,5,\dots \\ \frac{2}{1+n}+\frac{2}{1-n} & \text{if } n=2,4,6,\dots \end{cases} - \frac{1}{8}\begin{cases} 0 & \text{if } n=1,3,5,\dots \\ \frac{2}{3+n}+\frac{2}{3-n} & \text{if } n=2,4,6,\dots \end{cases} \\
&= \begin{cases} 0 & \text{if } n=1,3,5,\dots \\ \frac{3}{4}(\frac{1}{1+n}+\frac{1}{1-n})-\frac{1}{4}(\frac{1}{3+n}+\frac{1}{3-n}) & \text{if } n=2,4,6,\dots \end{cases} \\
&= \begin{cases} 0 & \text{if } n=1,3,5,\dots \\ \frac{12}{n^4-10n^2+9} & \text{if } n=2,4,6,\dots \end{cases}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
A_0 &= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} 1+\sin^3(x)\,dx \\
&= \frac{2}{\pi}\left(\pi+\frac{4}{3}\right) \\
&= 2+\frac{4}{3\pi}
\end{aligned}
$$

and

$$
\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} (1+\sin^3(x))\cos(nx)\,dx \\
&= \frac{2}{\pi}\begin{cases} 0 & \text{if } n=1,3,5,\dots \\ \frac{12}{n^4-10n^2+9} & \text{if } n=2,4,6,\dots \end{cases} \\
&= \begin{cases} 0 & \text{if } n=1,3,5,\dots \\ \frac{24}{\pi}\frac{1}{n^4-10n^2+9} & \text{if } n=2,4,6,\dots \end{cases}.
\end{aligned}
$$

Therefore, the formal solution is

$$\begin{aligned}
u(x,t) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-n^2kt} \cos\left(\frac{n\pi}{L}x\right) \\
&= \frac{1}{2}\left(2 + \frac{4}{3\pi}\right) + \sum_{n=2,4,6,\ldots} \frac{24}{\pi} \frac{1}{n^4 - 10n^2 + 9} e^{-n^2kt} \cos\left(\frac{n\pi}{L}x\right) \\
&= 1 + \frac{2}{3\pi} + \frac{24}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n)^4 - 10(2n)^2 + 9} e^{-(2n)^2kt} \cos\left(\frac{(2n)\pi}{L}x\right) \\
&= \boxed{1 + \frac{2}{3\pi} + \frac{24}{\pi} \sum_{n=1}^{\infty} \frac{1}{16n^4 - 40n^2 + 9} e^{-4n^2kt} \cos\left(\frac{2n\pi}{L}x\right)},
\end{aligned}$$

as desired. □

(c) Find $\lim_{t\to\infty} u(x,t)$ for all $0 < x < \pi$, and explain the physical interpretation of your result.

Solution. For all $0 < x < \pi$, we compute

$$\begin{aligned}
\lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty} \left(\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-12n^2t} \cos(nx)\right) \\
&= \lim_{t\to\infty} \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \lim_{t\to\infty} e^{-12n^2t} \cos(nx) \\
&= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cdot 0 \cdot \cos(nx) \\
&= \boxed{\frac{A_0}{2}}.
\end{aligned}$$

Physical interpretation (copied from the textbook solution manual)**:** We have shown that the quantity $\int_0^L u(x)\,dx$ is conserved in a one-dimensional insulated rod. The quantity $ku_x(x,t)$ measures the heat flux at a point x and time t. The homogeneous Neumann condition amounts to stating that there is zero flux at the rod's ends. Since there are no heat sources either (the equation is homogeneous), the temperature's gradient decays; therefore the temperature converges to a constant, such that the total stored energy is the same as the initial energy. □

5.6. (a) Using the method of separation of variables, find a formal solution of the problem

$$\begin{aligned}
u_t - ku_{xx} &= 0 && \text{if } 0 < x < L, t > 0, \\
u(0,t) = u(2\pi,t),\ u_x(0,t) &= u_x(2\pi,t) && \text{if } t \geq 0, \\
u(x,0) &= f(x) && \text{if } 0 \leq x \leq L.
\end{aligned}$$

Solution. We want to find a solution of the form

$$u(x,t) = X(x)T(t).$$

Our partial derivatives are

$$\begin{aligned}
u_t(x,t) &= X(x)T_t(t), \\
u_{xx}(x,t) &= X_{xx}(x)T(t).
\end{aligned}$$

So the partial differential equation

$$u_t - ku_{xx} = 0$$

becomes

$$X(x)T_t(t) - kX_{xx}(x)T(t) = 0,$$

which we can algebraically rearrange to write

$$\frac{X_{xx}(x)}{X(x)} = \frac{T_t(t)}{kT(t)} = -\lambda,$$

where λ is a constant in both x and t. This produces the system of two ordinary differential equations

$$\begin{aligned}
\frac{d^2X}{dx^2} + \lambda X &= 0 \\
\frac{dT}{dt} + \lambda kT &= 0.
\end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$X(x) = \begin{cases} C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0, \\ C_1 x + C_2 & \text{if } \lambda = 0, \\ C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) & \text{if } \lambda > 0, \end{cases}$$
$$T(t) = De^{-\lambda k t}$$

where C_1, C_2, D are constants. Now, the boundary conditions

$$\begin{aligned} u(0,t) &= u(2\pi,t), \\ u_x(0,t) &= u_x(2\pi,t) \end{aligned}$$

are equivalent to

$$\begin{aligned} X(0)T(t) &= X(2\pi)T(t), \\ \frac{d}{dx}X(0)T(t) &= \frac{d}{dx}X(2\pi)T(t), \end{aligned}$$

which imply either $T(t) = 0$ or $X(0) = X(2\pi)$ and $\frac{d}{dx}X(0) = \frac{d}{dx}X(2\pi)$. If $T(t) = 0$, then we would have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= X(x)0 \\ &= 0, \end{aligned}$$

which would be a trivial solution. But we are really interested in finding a nontrivial solution. So we should assume

$$\begin{aligned} X(0) &= X(2\pi), \\ \frac{d}{dx}X(0) &= \frac{d}{dx}X(2\pi), \end{aligned}$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then we have

  $$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}, \\ X(0) &= X(2\pi), \end{aligned}$$

  which implies $C_1 + C_2 = C_1 e^{2\pi\sqrt{-\lambda}} + C_2 e^{-2\pi\sqrt{-\lambda}}$. We also have

  $$\begin{aligned} \frac{d}{dx}X(x) &= \sqrt{-\lambda}(C_1 e^{\sqrt{-\lambda}x} - C_2 e^{-\sqrt{-\lambda}x}), \\ \frac{d}{dx}X(0) &= \frac{d}{dx}X(2\pi), \end{aligned}$$

  which implies $C_1 - C_2 = C_1 e^{2\pi\sqrt{-\lambda}} - C_2 e^{-2\pi\sqrt{-\lambda}}$. Now we will solve for the constants C_1, C_2. We have formulated the linear system

  $$\begin{aligned} C_1 + C_2 &= C_1 e^{2\pi\sqrt{-\lambda}} + C_2 e^{-2\pi\sqrt{-\lambda}}, \\ C_1 - C_2 &= C_1 e^{2\pi\sqrt{-\lambda}} - C_2 e^{-2\pi\sqrt{-\lambda}}, \end{aligned}$$

  and we can algebraically rearrange each equation in the system to write

  $$\begin{aligned} C_1(1 - e^{2\pi\sqrt{-\lambda}}) &= -C_2(1 - e^{-2\pi\sqrt{-\lambda}}), \\ C_1(1 - e^{2\pi\sqrt{-\lambda}}) &= C_2(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

  We can combine the two equations in the system to deduce

  $$\begin{aligned} C_1(1 - e^{2\pi\sqrt{-\lambda}}) &= -C_2(1 - e^{-2\pi\sqrt{-\lambda}}) \\ &= -C_1(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

  Since we are currently in the case $\lambda < 0$, we have $1 - e^{2\pi\sqrt{-\lambda}} \neq 0$, and so we can divide $1 - e^{2\pi\sqrt{-\lambda}}$ from both sides of our previous equation to conclude $C_1 = -C_1$, or $C_1 = 0$. Likewise, we can combine the two equations in the system to deduce

  $$\begin{aligned} C_2(1 - e^{-2\pi\sqrt{-\lambda}}) &= C_1(1 - e^{2\pi\sqrt{-\lambda}}) \\ &= -C_2(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

Since we are currently in the case $\lambda < 0$, we have $1 - e^{-2\pi\sqrt{-\lambda}} \neq 0$, and so we can divide $1 - e^{-2\pi\sqrt{-\lambda}}$ from both sides of our previous equation to conclude $C_2 = -C_2$, or $C_2 = 0$. So we have

$$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \\ &= 0e^{\sqrt{-\lambda}x} + 0e^{-\sqrt{-\lambda}x} \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= 0T(t) \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 2: Suppose $\lambda = 0$. Then we have

$$\begin{aligned} X(x) &= C_1 x + C_2, \\ X(0) &= X(2\pi), \end{aligned}$$

which implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 x + C_2 \\ &= C_1 \cdot 0 + C_2 \\ &= C_2. \end{aligned}$$

The derivative is

$$\begin{aligned} \frac{d}{dx}X(x) &= \frac{d}{dx}(C_2) \\ &= 0, \end{aligned}$$

which clearly satisfies $\frac{d}{dx}X(0) = 0 = \frac{d}{dx}X(2\pi)$. Therefore, if we write $\frac{A_0}{2} = C_2 D$, then we have

$$\begin{aligned} u_0(x,t) &= X(x)T(t) \\ &= C_2 D e^{-\lambda k(0)} \\ &= C_2 D \\ &= \frac{A_0}{2}, \end{aligned}$$

which is a nontrivial solution.

- Case 3: Suppose $\lambda > 0$. Then we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x), \\ X(0) &= X(2\pi), \end{aligned}$$

which implies

$$C_1 = C_1 \cos(2\pi\sqrt{\lambda}) + C_2 \sin(2\pi\sqrt{\lambda}). \tag{1}$$

We also have

$$\begin{aligned} \frac{d}{dx}X(x) &= \sqrt{\lambda}(-C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x)), \\ \frac{d}{dx}X(0) &= \frac{d}{dx}X(2\pi), \end{aligned}$$

which implies

$$C_2 = -C_1 \sin(2\pi\sqrt{\lambda}) + C_2 \cos(2\pi\sqrt{\lambda}). \tag{2}$$

Now, we claim that, if either $\sin(2\pi\sqrt{\lambda}) \neq 0$ or $\cos(2\pi\sqrt{\lambda}) \neq 1$, then we have $C_1 = 0$ and $C_2 = 0$.

  - Subcase 1: Suppose $\sin(2\pi\sqrt{\lambda}) \neq 0$. Multiply both sides of (1) by $-\cos(2\pi\sqrt{\lambda})$ and both sides of (2) by $\sin(2\pi\sqrt{\lambda})$ to obtain

$$\begin{aligned} -C_1 \cos(2\pi\sqrt{\lambda}) &= -C_1 \cos^2(2\pi\sqrt{\lambda}) - C_2 \sin(2\pi\sqrt{\lambda})\cos(2\pi\sqrt{\lambda}), \\ C_2 \sin(2\pi\sqrt{\lambda}) &= -C_1 \sin^2(2\pi\sqrt{\lambda}) + C_2 \cos(2\pi\sqrt{\lambda})\sin(2\pi\sqrt{\lambda}), \end{aligned}$$

from which we can add up both sides of the two equations to get

$$-C_1 \cos(2\pi\sqrt{\lambda}) + C_2 \sin(2\pi\sqrt{\lambda}) = -C_1. \tag{3}$$

We equate (1) and (3) to get

$$\cancel{C_1 \cos(2\pi\sqrt{\lambda})} - C_2 \sin(2\pi\sqrt{\lambda}) = \cancel{C_1 \cos(2\pi\sqrt{\lambda})} + C_2 \sin(2\pi\sqrt{\lambda}),$$

which simplifies to

$$-C_2 \cancel{\sin(2\pi\sqrt{\lambda})} = C_2 \cancel{\sin(2\pi\sqrt{\lambda})}.$$

Since we assumed $\sin(2\pi\sqrt{\lambda}) \neq 0$, we can divide both sides by $\sin(2\pi\sqrt{\lambda})$ to get $-C_2 = C_2$, which means $C_2 = 0$. Substitute $C_2 = 0$ into (2) to obtain

$$0 = -C_1 \sin(2\pi\sqrt{\lambda}),$$

which implies $C_1 = 0$ because, once again, we assumed $\sin(2\pi\sqrt{\lambda}) \neq 0$.

– Subcase 2: Suppose $\cos(2\pi\sqrt{\lambda}) \neq 1$. Then we can rewrite (1) and (2) as

$$C_1(1 - \cos(2\pi\sqrt{\lambda})) = C_2 \sin(2\pi\sqrt{\lambda}), \tag{4}$$

$$C_2(1 - \cos(2\pi\sqrt{\lambda})) = -C_1 \sin(2\pi\sqrt{\lambda}),. \tag{5}$$

Multiply both sides of (4) by C_1 and both sides of (5) by C_2 to obtain

$$\begin{aligned} C_1^2(1 - \cos(2\pi\sqrt{\lambda})) &= C_1C_2 \sin(2\pi\sqrt{\lambda}), \\ C_2^2(1 - \cos(2\pi\sqrt{\lambda})) &= -C_1C_2 \sin(2\pi\sqrt{\lambda}), \end{aligned}$$

from which we can add up both sides of the two equations to get

$$(C_1^2 + C_2^2)(1 - \cos(2\pi\sqrt{\lambda})) = 0.$$

Since we assumed $\cos(2\pi\sqrt{\lambda}) \neq 1$, we must conclude $C_1^2 + C_2^2 = 0$, which forces $C_1 = 0$ and $C_2 = 0$.

So we have proved our claim. Now that we have established our claim, we would have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \\ &= 0 \cos(\sqrt{\lambda}x) + 0 \sin(\sqrt{\lambda}x) \\ &= 0, \end{aligned}$$

which would imply that $u(x,t) = X(x)T(t)$ is a trivial solution. Therefore, to find a nontrivial solution for this case, we should assume both

$$\begin{aligned} \sin(2\pi\sqrt{\lambda}) &= 0, \\ 1 - \cos(2\pi\sqrt{\lambda}) &= 0, \end{aligned}$$

which imply $2\pi\sqrt{\lambda} = 2n\pi$, or equivalently

$$\lambda_n = \lambda = n^2,$$

and so we have

$$\begin{aligned} X_n(x) &= C_{1,n} \cos(\sqrt{\lambda_n}x) + C_{2,n} \sin(\sqrt{\lambda_n}x) \\ &= C_{1,n} \cos(\sqrt{n^2}x) + C_{2,n} \sin(\sqrt{n^2}x) \\ &= C_{1,n} \cos(nx) + C_{2,n} \sin(nx) \end{aligned}$$

and

$$\begin{aligned} T_n(t) &= D_n e^{-\lambda_n kt} \\ &= D_n e^{-n^2 kt} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Therefore, if we write $A_n = C_{1,n}D_n$ and $B_n = C_{2,n}D_n$, then we have

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ &= (C_{1,n} \cos(nx) + C_{2,n} \sin(nx))(D_n e^{-(\frac{n\pi}{L})^2 kt}) \\ &= e^{-n^2 kt}(C_{1,n}D_n \cos(nx) + C_{2,n}D_n \sin(nx)) \\ &= e^{-n^2 kt}(A_n \cos(nx) + B_n \sin(nx)) \end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial solution, as desired.

We recall that an addition of solutions is again a solution. So that means, as we have established already that each $u_n(x,t)$ is a nontrivial solution for $n = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned} u(x,t) &= u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) \\ &= \boxed{\frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-n^2 kt}(A_n \cos(nx) + B_n \sin(nx))}, \end{aligned}$$

where

$$\begin{aligned} A_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx, \\ A_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\, dx, \\ B_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\, dx \end{aligned}$$

for $n = 1, 2, 3, \ldots$ are the Fourier coefficients, is also a solution of the problem. □

(b) Find $\lim_{t\to\infty} u(x,t)$ for all $0 < x < 2\pi$, and explain the physical interpretation of your result.

Solution. For all $0 < x < \pi$, we compute

$$\begin{aligned} \lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\left(\frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-n^2 kt}(A_n\cos(nx) + B_n\sin(nx))\right) \\ &= \lim_{t\to\infty}\frac{A_0}{2} + \sum_{n=1}^{\infty}\lim_{t\to\infty} e^{-n^2kt}(A_n\cos(nx) + B_n\sin(nx)) \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty}\lim_{t\to\infty} 0(A_n\cos(nx) + B_n\sin(nx)) \\ &= \boxed{\frac{A_0}{2}}. \end{aligned}$$

Physical interpretation: We have shown that the quantity $\int_0^L u(x)\, dx$ is conserved in a one-dimensional insulated rod. The quantity $ku_x(x,t)$ measures the heat flux at a point x and time t. The periodic Dirichlet condition amounts to stating that the amount of thermal energy is the same at the rod's ends, and the periodic Neumann condition amounts to stating that the flux is the same at the rod's ends. Since there are no heat sources either (the equation is homogeneous), the temperature's gradient decays; therefore the temperature converges to a constant, such that the total stored energy is the same as the initial energy. □

5.7. Solve the problem

$$\begin{aligned} u_t - ku_{xx} &= A\cos(\alpha t) && \text{if } 0 < x < 1, t > 0, \\ u_x(0,t) = u_x(1,t) &= 0 && \text{if } t \geq 0, \\ u(x,0) &= 1 + \cos^2(\pi x) && \text{if } 0 \leq x \leq 1. \end{aligned}$$

Solution. First, we need to find all the eigenvalues and eigenfunctions of the homogeneous problem

$$\begin{aligned} u_t - ku_{xx} &= 0 && \text{if } 0 < x < 1, t > 0, \\ u_x(0,t) = u_x(1,t) &= 0 && \text{if } t \geq 0, \\ u(x,0) &= 1 + \cos^2(\pi x) && \text{if } 0 \leq x \leq 1. \end{aligned}$$

To do this, we can proceed as we did in the method of separation of variables by writing

$$u(x,t) = X(x)T(t).$$

Our partial derivatives are

$$\begin{aligned} u_t(x,t) &= X(x)T_t(t), \\ u_{xx}(x,t) &= X_{xx}(x)T(t). \end{aligned}$$

So the partial differential equation

$$u_t - 4u_{xx} = 0$$

becomes

$$X(x)T_t(t) - kX_{xx}(x)T(t) - hX(x)T(t) = 0,$$

which we can algebraically rearrange to write

$$\frac{X_{xx}(x)}{X(x)} = \frac{T_t(t)}{kT(t)} = -\lambda,$$

where λ is a constant in both x and t. This produces the system of two ordinary differential equations

$$\begin{aligned} \frac{d^2X}{dx^2} + \lambda X &= 0 \\ \frac{dT}{dt} + k\lambda T &= 0. \end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$\begin{aligned} X(x) &= \begin{cases} C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0, \\ C_1 x + C_2 & \text{if } \lambda = 0, \\ C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) & \text{if } \lambda > 0, \end{cases} \\ T(t) &= De^{-k\lambda t} \end{aligned}$$

where C_1, C_2, D are constants. Now, the boundary conditions

$$u_x(0,t) = u_x(1,t) = 0$$

are equivalent to

$$\begin{aligned} \frac{d}{dx}X(0)T(t) &= 0, \\ \frac{d}{dx}X(1)T(t) &= 0, \end{aligned}$$

which imply either $T(t) = 0$ or $\frac{d}{dx}X(0) = \frac{d}{dx}X(1) = 0$. If $T(t) = 0$, then we would have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= X(x)0 \\ &= 0, \end{aligned}$$

which would be a trivial solution. So we should assume

$$\frac{d}{dx}X(0) = \frac{d}{dx}X(1) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then we have

  $$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}, \\ X(0) &= 0, \end{aligned}$$

  which implies $C_1 + C_2 = 0$, or $C_2 = -C_1$. So we have

  $$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \\ &= C_1 e^{\sqrt{-\lambda}x} - C_1 e^{-\sqrt{-\lambda}x} \\ &= C_1(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}). \end{aligned}$$

  We notice $e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L} \neq 0$ unless $\lambda = 0$. This means

  $$\begin{aligned} X(x) &= C_1(e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}), \\ X(L) &= 0 \end{aligned}$$

  implies $C_1 = 0$, and so we have

  $$\begin{aligned} X(x) &= C_1(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}) \\ &= 0(e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}) \\ &= 0, \end{aligned}$$

  which would mean u is a trivial solution. Therefore, the problem has no negative eigenvalues.

- Case 2: Suppose $\lambda = 0$. Then we have

$$\begin{aligned} X(x) &= C_1 x + C_2, \\ \frac{d}{dx} X(0) &= 0, \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dx} X(x) &= C_1, \\ \frac{d}{dx} X(0) &= 0, \end{aligned}$$

which implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 x + C_2 \\ &= 0x + C_2 \\ &= C_2, \end{aligned}$$

which already satisfies $\frac{d}{dx} X(1) = 0$. Therefore, if we write $\frac{A_0}{2} = C_2 D$, then we have

$$\begin{aligned} u_0(x,t) &= X_0(x) T_0(t) \\ &= C_2 D e^{-\lambda \cdot 0} \\ &= C_2 D \\ &= \frac{A_0}{2}, \end{aligned}$$

which is a nontrivial solution.

- Case 3: Suppose $\lambda > 0$. Then we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x), \\ \frac{d}{dx} X(0) &= 0, \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dx} X(x) &= \sqrt{\lambda}(-C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x)), \\ \frac{d}{dx} X(0) &= 0, \end{aligned}$$

which implies $C_2 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x) \\ &= C_1 \cos(\sqrt{\lambda} x) + 0 \sin(\sqrt{\lambda} x) \\ &= C_1 \cos(\sqrt{\lambda} x). \end{aligned}$$

Next, we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda} x), \\ \frac{d}{dx} X(1) &= 0 \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dx} X(x) &= -\sqrt{\lambda} C_1 \sin(\sqrt{\lambda} x), \\ \frac{d}{dx} X(1) &= 0 \end{aligned}$$

implies either $C_1 = 0$ or $\sin(\sqrt{\lambda}) = 0$. But $C_2 = 0$ (with $C_1 = 0$) implies $X(x) = 0$ and that $u(x,t)$ would be a trivial solution. So we should assume $\sqrt{\lambda} = n\pi$, or equivalently the eigenvalues

$$\lambda_n = \lambda = (n\pi)^2,$$

with the corresponding eigenfunctions

$$\begin{aligned} X_n(x) &= C_{1,n} \cos(\sqrt{\lambda_n} x) \\ &= C_{1,n} \cos(\sqrt{(n\pi)^2} x) \\ &= C_{1,n} \cos(n\pi x), \end{aligned}$$

as desired.

From the three cases above, we see that the problem has the zero eigenvalue $\lambda = 0$ and its corresponding eigenfunction $X_0(x) = C_2$, as well as positive eigenvalues $\lambda_n = (n\pi)^2$ and their corresponding eigenfunctions $X_n(x) = C_{1,n}\cos(n\pi x)$ (or just $X_n(x) = \cos(n\pi x)$; these two eigenfunctions are the same up to a scaling factor). We will now use the method of eigenfunction expansion. Based on our eignefunction $X_n(x) = \cos(n\pi x)$, we can represent, for any fixed t, our solution as

$$u(x,t) = \frac{1}{2}T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(n\pi x),$$

where $T_n(t)$ for $n = 1, 2, 3, \dots$ are the time-dependent Fourier coefficients. Our derivatives are

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\left(\frac{1}{2}T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(n\pi x)\right) \\ &= \frac{1}{2}T_0'(t) + \sum_{n=1}^{\infty} T_n'(t)\cos(n\pi x) \end{aligned}$$

and

$$\begin{aligned} u_{xx}(x,t) &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}T_0'(t) + \sum_{n=1}^{\infty} T_n'(t)\cos(n\pi x)\right) \\ &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}T_0'(t)\right) + \sum_{n=1}^{\infty} T_n(t)\frac{\partial^2}{\partial x^2}\cos(n\pi x) \\ &= 0 + \sum_{n=1}^{\infty} -(n\pi)^2 T_n(t)\cos(n\pi x) \\ &= \sum_{n=1}^{\infty} -(n\pi)^2 T_n(t)\cos(n\pi x). \end{aligned}$$

So the nonhomogeneous partial differential equation

$$u_t - 4u_{xx} = A\cos(\alpha t)$$

becomes

$$\left(\frac{1}{2}T_0'(t) + \sum_{n=1}^{\infty} T_n'(t)\cos(n\pi x)\right) - k\left(\sum_{n=1}^{\infty} -(n\pi)^2 T_n(t)\cos(n\pi x)\right) = A\cos(\alpha t),$$

or equivalently

$$\frac{1}{2}T_0'(t) + \sum_{n=1}^{\infty}(T_n'(t) + k(n\pi)^2 T_n(t))\cos(n\pi x) = A\cos(\alpha t) + 0\cos(n\pi x).$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to obtain the ordinary differential equations

$$\begin{aligned} \frac{1}{2}T_0'(t) - A\cos(\alpha t) &= 0, \\ T_n'(t) + k(n\pi)^2 T_n(t) &= 0, \end{aligned}$$

whose general solutions are, respectively,

$$\begin{aligned} T_0(t) &= \begin{cases} \frac{2A}{\alpha}\sin(\alpha t) + B_0 & \text{if } \alpha \neq 0 \\ 2At + B_0 & \text{if } \alpha = 0, \end{cases} \\ T_n(t) &= B_n e^{-k(n\pi)^2 t}, \end{aligned}$$

where A_0 and A_n for $n = 1, 2, 3, \dots$ are the Fourier coefficients. Therefore, our solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(n\pi x) \\ &= \frac{1}{2}\begin{cases} \frac{2A}{\alpha}\sin(\alpha t) + B_0 & \text{if } \alpha \neq 0 \\ 2At + B_0 & \text{if } \alpha = 0 \end{cases} + \sum_{n=1}^{\infty} B_n e^{-k(n\pi)^2 t}\cos(n\pi x) \\ &= \begin{cases} \frac{A}{\alpha}\sin(\alpha t) + \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n e^{-k(n\pi)^2 t}\cos(n\pi x) & \text{if } \alpha \neq 0 \\ At + \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n e^{-k(n\pi)^2 t}\cos(n\pi x) & \text{if } \alpha = 0 \end{cases}. \end{aligned}$$

Now, we can use the given initial conditions to write

$$\begin{aligned} u(x,0) &= 1+\cos^2(\pi x) \\ &= 1+\frac{1}{2}(1+\cos(2\pi x)) \\ &= \frac{3}{2}+\frac{1}{2}\cos(2\pi x), \end{aligned}$$

where in the last step above we have employed the double-angle trigonometric identity $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$. Also, at $t=0$, our solution becomes

$$\begin{aligned} u(x,0) &= \frac{B_0}{2}+\sum_{n=1}^{\infty} B_n\cos(n\pi x) \\ &= \frac{B_0}{2}+B_1\cos(\pi x)+B_2\cos(2\pi x)+\sum_{n=3}^{\infty} B_n\cos(n\pi x). \end{aligned}$$

Both our expressions of $u(x,0)$ yield

$$\frac{B_0}{2}+B_1\cos(\pi x)+B_2\cos(2\pi x)+\sum_{n=3}^{\infty} B_n\cos(n\pi x) = \frac{3}{2}+\frac{1}{2}\cos(2\pi x).$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to obtain the Fourier coefficients

$$\begin{aligned} B_0 &= 3, \\ B_2 &= \frac{1}{2}, \\ B_n &= 0 \end{aligned}$$

for $n=1$ and for $n=3,4,5,\ldots$. Therefore, our formal solution is

$$\begin{aligned} u(x,t) &= \begin{cases} \frac{A}{\alpha}\sin(\alpha t)+\frac{B_0}{2}+\sum_{n=1}^{\infty} B_n e^{-k(n\pi)^2 t}\cos(n\pi x) & \text{if } \alpha\neq 0 \\ At+\frac{B_0}{2}+\sum_{n=1}^{\infty} B_n e^{-k(n\pi)^2 t}\cos(n\pi x) & \text{if } \alpha=0 \end{cases} \\ &= \begin{cases} \frac{A}{\alpha}\sin(\alpha t)+\frac{B_0}{2}+B_1 e^{-k(1\pi)^2 t}\cos(1\pi x)+B_2 e^{-k(2\pi)^2 t}\cos(2\pi x)+\sum_{n=3}^{\infty} B_n e^{-k(n\pi)^2 t}\cos(n\pi x) & \text{if } \alpha\neq 0 \\ At+\frac{B_0}{2}+B_1 e^{-k(1\pi)^2 t}\cos(1\pi x)+B_2 e^{-k(2\pi)^2 t}\cos(2\pi x)+\sum_{n=3}^{\infty} B_n e^{-k(n\pi)^2 t}\cos(n\pi x) & \text{if } \alpha=0 \end{cases} \\ &= \begin{cases} \frac{A}{\alpha}\sin(\alpha t)+\frac{3}{2}+0e^{-k(1\pi)^2 t}\cos(1\pi x)+\frac{1}{2}e^{-k(2\pi)^2 t}\cos(2\pi x)+\sum_{n=3}^{\infty} 0e^{-k(n\pi)^2 t}\cos(n\pi x) & \text{if } \alpha\neq 0 \\ At+\frac{3}{2}+0e^{-k(1\pi)^2 t}\cos(1\pi x)+\frac{1}{2}e^{-k(2\pi)^2 t}\cos(2\pi x)+\sum_{n=3}^{\infty} 0e^{-k(n\pi)^2 t}\cos(n\pi x) & \text{if } \alpha=0 \end{cases} \\ &= \boxed{\begin{cases} \frac{A}{\alpha}\sin(\alpha t)+\frac{1}{2}e^{-4k\pi^2 t}\cos(2\pi x)+\frac{3}{2} & \text{if } \alpha\neq 0 \\ At+\frac{1}{2}e^{-4k\pi^2 t}\cos(2\pi x)+\frac{3}{2} & \text{if } \alpha=0 \end{cases}}, \end{aligned}$$

as desired. □

5.8. Consider the problem

$$\begin{aligned} u_t-u_{xx} &= e^{-t}\sin(3x) && \text{if } 0<x<\pi, t>0, \\ u(0,t)=u(\pi,t) &= 0 && \text{if } t\geq 0, \\ u(x,0) &= f(x) && \text{if } 0\leq x\leq \pi. \end{aligned}$$

(a) Solve the problem using the method of eigenfunction expansion.

Solution. First, we need to find all the eigenvalues and eigenfunctions of the homogeneous problem

$$\begin{aligned} u_t-u_{xx} &= 0 && \text{if } 0<x<L, t>0, \\ u(0,t)=u(\pi,t) &= 0 && \text{if } t\geq 0, \\ u(x,0) &= f(x) && \text{if } 0\leq x\leq L. \end{aligned}$$

To do this, we can proceed as we did in the method of separation of variables by writing

$$u(x,t) = X(x)T(t).$$

Our partial derivatives are

$$\begin{aligned} u_t(x,t) &= X(x)T_t(t), \\ u_{xx}(x,t) &= X_{xx}(x)T(t). \end{aligned}$$

So the partial differential equation

$$u_t - u_{xx} = 0$$

becomes

$$X(x)T_t(t) - X_{xx}(x)T(t) = 0,$$

which we can algebraically rearrange to write

$$\frac{X_{xx}(x)}{X(x)} = \frac{T_t(t)}{T(t)} = -\lambda,$$

where λ is a constant in both x and t. This produces the system of two ordinary differential equations

$$\begin{aligned}\frac{d^2X}{dx^2} + \lambda X &= 0 \\ \frac{dT}{dt} + \lambda T &= 0.\end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$\begin{aligned}X(x) &= \begin{cases} C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0, \\ C_1 x + C_2 & \text{if } \lambda = 0, \\ C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) & \text{if } \lambda > 0, \end{cases} \\ T(t) &= De^{-\lambda t}\end{aligned}$$

where C_1, C_2, D are constants. Now, the boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

are equivalent to

$$\begin{aligned}X(0)T(t) &= 0, \\ X(\pi)T(t) &= 0,\end{aligned}$$

which imply either $T(t) = 0$ or $X(0) = X(\pi) = 0$. If $T(t) = 0$, then we would have

$$\begin{aligned}u(x,t) &= X(x)T(t) \\ &= X(x)0 \\ &= 0,\end{aligned}$$

which would be a trivial solution. So we should assume

$$X(0) = X(\pi) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then we have

  $$\begin{aligned}X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}, \\ X(0) &= 0,\end{aligned}$$

  which implies $C_1 + C_2 = 0$, or $C_2 = -C_1$. So we have

  $$\begin{aligned}X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \\ &= C_1 e^{\sqrt{-\lambda}x} - C_1 e^{-\sqrt{-\lambda}x} \\ &= C_1(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}).\end{aligned}$$

  We notice $e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L} \neq 0$ unless $\lambda = 0$. This means

  $$\begin{aligned}X(x) &= C_1(e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}), \\ X(L) &= 0\end{aligned}$$

  implies $C_1 = 0$, and so we have

  $$\begin{aligned}X(x) &= C_1(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}) \\ &= 0(e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}) \\ &= 0,\end{aligned}$$

  which would mean u is a trivial solution. Therefore, the problem has no negative eigenvalues.

- Case 2: Suppose $\lambda = 0$. Then we have

$$\begin{aligned} X(x) &= C_1 x + C_2, \\ X(0) &= 0, \end{aligned}$$

which implies $C_2 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 x + C_2 \\ &= C_1 x + 0 \\ &= C_1 x. \end{aligned}$$

And we have

$$\begin{aligned} X(x) &= C_1 x, \\ X(\pi) &= 0, \end{aligned}$$

which implies $C_1 = 0$. So we have

$$\begin{aligned} X(x) &= C_1 x \\ &= 0x \\ &= 0, \end{aligned}$$

which would mean u is a trivial solution. Therefore, 0 is not an eigenvalue of the problem.

- Case 3: Suppose $\lambda > 0$. Then we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x), \\ X(0) &= 0, \end{aligned}$$

which implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x) \\ &= 0 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x) \\ &= C_2 \sin(\sqrt{\lambda} x). \end{aligned}$$

Next,

$$\begin{aligned} X(x) &= C_2 \sin(\sqrt{\lambda} x), \\ X(\pi) &= 0 \end{aligned}$$

implies either $C_2 = 0$ or $\sin(\sqrt{\lambda}\pi) = 0$. But $C_2 = 0$ (with $C_1 = 0$) implies $X(x) = 0$ and that $u(x,t)$ would be a trivial solution. So we should assume $\sqrt{\lambda}\pi = n\pi$, or equivalently the eigenvalues

$$\lambda_n = \lambda = n^2,$$

with the corresponding eigenfunctions

$$\begin{aligned} X_n(x) &= C_{2,n} \sin(\sqrt{\lambda_n} x) \\ &= C_{2,n} \sin(\sqrt{n^2} x) \\ &= C_{2,n} \sin(nx), \end{aligned}$$

as desired.

From the three cases above, we see that the problem only has positive eigenvalues $\lambda_n = n^2$ and their corresponding eigenfunctions $X_n(x) = C_{2,n} \sin(nx)$. We will now use the method of eigenfunction expansion. We can represent, for any fixed t, our solution as

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(nx),$$

where $T_n(t)$ for $n = 1, 2, 3, \ldots$ are the time-dependent Fourier coefficients. (Note that, unlike Section 5.4 in the textbook, we do not have the term $\frac{1}{2}T_0(t)$ in our representation of $u(x,t)$ because 0 is not an eigenvalue of this problem.) Our derivatives are

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} T_n(t) \sin(nx)\right) \\ &= \sum_{n=1}^{\infty} T_n'(t) \sin(nx) \end{aligned}$$

and

$$\begin{aligned} u_{xx}(x,t) &= \frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} T_n(t)\sin(nx)\right) \\ &= \sum_{n=1}^{\infty} T_n(t)\frac{\partial^2}{\partial x^2}(\sin(nx)) \\ &= \sum_{n=1}^{\infty} -n^2 T_n(t)\sin(nx). \end{aligned}$$

So the nonhomogeneous partial differential equation

$$u_t - u_{xx} = e^{-t}\sin(3x)$$

becomes

$$\sum_{n=1}^{\infty} T_n'(t)\sin(nx) - \sum_{n=1}^{\infty} -n^2 T_n(t)\sin(nx) = e^{-t}\sin(3x),$$

or equivalently

$$\sum_{n=1}^{\infty}(T_n'(t) + n^2 T_n(t))\sin(nx) = e^{-t}\sin(3x).$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to obtain the ordinary differential equations

$$T_n'(t) + n^2 T_n(t) = \begin{cases} e^{-t} & \text{if } n = 3, \\ 0 & \text{if } n = 1,2 \text{ and } n = 4,5,6,\dots, \end{cases}$$

whose general solutions are

$$T_n(t) = \begin{cases} B_3 e^{-3^2 t} + \frac{1}{8}e^{-t} & \text{if } n = 3, \\ B_n e^{-n^2 t} & \text{if } n = 1,2 \text{ and } n = 4,5,6,\dots., \end{cases}$$

where

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

for $n = 1,2,3,\dots$ are the Fourier coefficients. Therefore, our solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} T_n(t)\sin(nx) \\ &= T_1(t)\sin(1x) + T_2(t)\sin(2x) + T_3(t)\sin(3x) + \sum_{n=4}^{\infty} T_n(t)\sin(nx) \\ &= B_1 e^{-1^2 t}\sin(1x) + B_2 e^{-2^2 t}\sin(2x) + \left(B_3 e^{-3^2 t} + \frac{1}{8}e^{-t}\right)\sin(3x) + \sum_{n=4}^{\infty} B_n e^{-n^2 t}\sin(nx) \\ &= B_1 e^{-1^2 t}\sin(1x) + B_2 e^{-2^2 t}\sin(2x) + B_3 e^{-3^2 t}\sin(3x) + \sum_{n=4}^{\infty} B_n e^{-n^2 t}\sin(nx) + \frac{1}{8}e^{-t}\sin(3x) \\ &= \boxed{\sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin(nx) + \frac{1}{8}e^{-t}\sin(3x)}, \end{aligned}$$

as desired. □

(b) Find $u(x,t)$ for $f(x) = x\sin(x)$.

Solution. In order for

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin(nx) + \frac{1}{8}e^{-t}\sin(3x)$$

to satisfy $u(x,0) = x\sin(x)$, we need to compute

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

with $f(x) = x\sin(x)$. Employing the trigonometric identity $\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$ with $\alpha = x$ and $\beta = nx$ and the method of integration by parts, we have for $n = 1, 2, 3, \ldots$

$$\begin{aligned}
\int_0^\pi x\sin(x)\sin(nx)\,dx &= \int_0^\pi x\left(\frac{1}{2}(\cos(x-nx) - \cos(x+nx)\right)dx \\
&= \int_0^\pi x\cos((1-n)x)\,dx - \frac{1}{2}\int_0^\pi x\cos((1+n)x)\,dx \\
&= \frac{1}{2}\left(\frac{1}{1-n}x\sin((1-n)x) + \frac{1}{(1-n)^2}\cos((1-n)x)\right)\Bigg|_0^\pi \\
&\quad - \frac{1}{2}\left(\frac{1}{1+n}x\sin((1+n)x) + \frac{1}{(1+n)^2}\cos((1+n)x)\right)\Bigg|_0^\pi \\
&= \frac{\cos((1-n)\pi) - 1}{2(1-n)^2} - \frac{\cos((1+n)\pi) - 1}{2(1+n)^2} \\
&= \frac{(-1)^{1-n} - 1}{2(1-n)^2} - \frac{(-1)^{1+n} - 1}{2(1+n)^2} \\
&= \begin{cases} 0 & \text{if } n = 1, 3, 5, \ldots \\ \frac{1}{(1+n)^2} - \frac{1}{(1-n)^2} & \text{if } n = 2, 4, 6, \ldots. \end{cases}
\end{aligned}$$

To compute B_n for all $n = 1, 2, 3, \ldots$, first note

$$\int_0^\pi \sin(nx)\sin(mx)\,dx = \begin{cases} \frac{\pi}{2} & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases}$$

where $m = 1, 2, 3, \ldots$ is a parameter. If $n = 1, 2$ or $n = 4, 5, 6, \ldots$, then we have

$$\begin{aligned}
\int_0^\pi x\sin(x)\sin(nx)\,dx &= \int_0^\pi u(x,0)\sin(nx)\,dx \\
&= \int_0^\pi \left(\sum_{m=1}^\infty B_m\sin(mx) + \frac{1}{8}\sin(3x)\right)\sin(nx)\,dx \\
&= \sum_{m=1}^\infty B_m\int_0^\pi \sin(mx)\sin(nx)\,dx + \frac{1}{8}\int_0^\pi \sin(3x)\sin(nx)\,dx \\
&= B_n\int_0^\pi \sin^2(nx)\,dx + \frac{1}{8}\int_0^\pi \sin(3x)\sin(nx)\,dx \\
&= B_n\cdot\frac{\pi}{2} + \frac{1}{8}\cdot 0 \\
&= \frac{\pi}{2}B_n,
\end{aligned}$$

which implies

$$\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^\pi x\sin(x)\sin(nx)\,dx \\
&= \begin{cases} 0 & \text{if } n = 1 \text{ or if } n = 5, 7, 9, \ldots \\ \frac{2}{\pi}\left(\frac{1}{(1+n)^2} - \frac{1}{(1-n)^2}\right) & \text{if } n = 2, 4, 6, \ldots \end{cases}.
\end{aligned}$$

If $n = 3$, then we have

$$\begin{aligned}
\int_0^\pi x\sin(x)\sin(3x)\,dx &= \int_0^\pi u(x,0)\sin(3x)\,dx \\
&= \int_0^\pi \left(\sum_{m=1}^\infty B_m\sin(mx) + \frac{1}{8}\sin(3x)\right)\sin(3x)\,dx \\
&= \sum_{m=1}^\infty B_m\int_0^\pi \sin(mx)\sin(3x) + \frac{1}{8}\int_0^\pi \sin^2(3x)\,dx \\
&= B_3\int_0^\pi \sin^2(3x)\,dx + \frac{1}{8}\int_0^\pi \sin^2(3x)\,dx \\
&= \left(B_3 + \frac{1}{8}\right)\int_0^\pi \sin^2(3x)\,dx \\
&= \left(B_3 + \frac{1}{8}\right)\frac{\pi}{2},
\end{aligned}$$

which implies

$$
\begin{aligned}
B_3 &= \frac{2}{\pi}\int_0^{\pi} x\sin(x)\sin(3x)\,dx - \frac{1}{8} \\
&= \frac{2}{\pi}\cdot 0 - \frac{1}{8} \\
&= -\frac{1}{8}.
\end{aligned}
$$

Therefore, the formal solution is

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin(nx) + \frac{1}{8}e^{-t}\sin(3x) \\
&= B_1 e^{-1^2 t}\sin(1x) + B_2 e^{-2^2 t}\sin(2x) + B_3 e^{-3^2 t}\sin(3x) + \sum_{n=4}^{\infty} B_n e^{-n^2 t}\sin(nx) + \frac{1}{8}e^{-t}\sin(3x) \\
&= 0e^{-1^2 t}\sin(1x) + \frac{2}{\pi}\left(\frac{1}{(1+2)^2} - \frac{1}{(1-2)^2}\right)e^{-2^2 t}\sin(2x) - \frac{1}{8}e^{-3^2 t}\sin(3x) \\
&\quad + \frac{2}{\pi}\sum_{n=4,6,8,\ldots}\left(\frac{1}{(1+n)^2} - \frac{1}{(1-n)^2}\right)e^{-n^2 t}\sin(nx) + \frac{1}{8}e^{-t}\sin(3x) \\
&= \frac{2}{\pi}\sum_{n=2,4,6,\ldots}\left(\frac{1}{(1+n)^2} - \frac{1}{(1-n)^2}\right)e^{-n^2 t}\sin(nx) - \frac{1}{8}e^{-9t}\sin(3x) + \frac{1}{8}e^{-t}\sin(3x) \\
&= \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{(1+(2n))^2} - \frac{1}{(1-(2n))^2}\right)e^{-(2n)^2 t}\sin((2n)x) + \frac{1}{8}\sin(3x)(e^{-t} - e^{-9t}) \\
&= \boxed{\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{(1+2n)^2} - \frac{1}{(1-2n)^2}\right)e^{-4n^2 t}\sin(2nx) + \frac{1}{8}\sin(3x)(e^{-t} - e^{-9t})},
\end{aligned}
$$

as desired. □

(c) Show that the solution $u(x,t)$ is indeed a solution of the equation

$$u_t - u_{xx} = e^{-t}\sin(3x)$$

for all $0 < x < \pi$ and $t > 0$.

Solution. The solution of the problem from our solution to part (a) is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin(nx) + \frac{1}{8}e^{-t}\sin(3x).$$

Our partial derivatives are

$$
\begin{aligned}
u_t &= \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin(nx) + \frac{1}{8}e^{-t}\sin(3x)\right) \\
&= \sum_{n=1}^{\infty} B_n \frac{\partial}{\partial t}e^{-n^2 t}\sin(nx) + \frac{1}{8}\frac{\partial}{\partial t}e^{-t}\sin(3x) \\
&= \sum_{n=1}^{\infty} -B_n n^2 e^{n^2 t}\sin(nx) - \frac{1}{8}e^{-t}\sin(3x)
\end{aligned}
$$

and

$$
\begin{aligned}
u_{xx} &= \frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin(nx) + \frac{1}{8}e^{-t}\sin(3x)\right) \\
&= \sum_{n=1}^{\infty} B_n e^{-n^2 t}\frac{\partial^2}{\partial x^2}\sin(nx) + \frac{1}{8}e^{-t}\frac{\partial^2}{\partial x^2}\sin(3x) \\
&= \sum_{n=1}^{\infty} -B_n n^2 e^{-n^2 t}\sin(nx) - \frac{9}{8}e^{-t}\sin(3x)
\end{aligned}
$$

So we have

$$\begin{aligned}
u_t - u_{xx} &= \left(\cancel{\sum_{n=1}^{\infty} -B_n n^2 e^{n^2 t}} \sin(nx) - \frac{1}{8}e^{-t}\sin(3x)\right) - \left(\cancel{\sum_{n=1}^{\infty} -B_n n^2 e^{-n^2 t}} \sin(nx) - \frac{9}{8}e^{-t}\sin(3x)\right) \\
&= -\frac{1}{8}e^{-t}\sin(3x) + \frac{9}{8}e^{-t}\sin(3x) \\
&= e^{-t}\sin(3x),
\end{aligned}$$

as desired. □

5.9. Consider the problem

$$\begin{aligned}
u_t - u_{xx} - hu &= 0 && \text{if } 0 < x < \pi, t > 0, \\
u(0,t) = u(\pi,t) &= 0 && \text{if } t \geq 0, \\
u(x,0) &= x(\pi - x) && \text{if } 0 \leq x \leq \pi,
\end{aligned}$$

where h is a real constant.

(a) Solve the problem using the method of eigenfunction expansion.

Solution. First, we need to find all the eigenvalues and eigenfunctions of the homogeneous problem

$$\begin{aligned}
u_t - u_{xx} - hu &= 0 && \text{if } 0 < x < L, t > 0, \\
u(0,t) = u(\pi,t) &= 0 && \text{if } t \geq 0, \\
u(x,0) &= f(x) && \text{if } 0 \leq x \leq L.
\end{aligned}$$

To do this, we can proceed as we did in the method of separation of variables by writing

$$u(x,t) = X(x)T(t).$$

Our partial derivatives are

$$\begin{aligned}
u_t(x,t) &= X(x)T_t(t), \\
u_{xx}(x,t) &= X_{xx}(x)T(t).
\end{aligned}$$

So the partial differential equation

$$u_t - u_{xx} = 0$$

becomes

$$X(x)T_t(t) - X_{xx}(x)T(t) - hX(x)T(t) = 0,$$

or

$$X(x)T_t(t) - (X_{xx} + hX(x))T(t) = 0,$$

which we can algebraically rearrange to write

$$\frac{X_{xx}(x) + hX(x)}{X(x)} = \frac{T_t(t)}{T(t)} = -\lambda,$$

where λ is a constant in both x and t. This produces the system of two ordinary differential equations

$$\begin{aligned}
\frac{d^2X}{dx^2} + (\lambda + h)X &= 0 \\
\frac{dT}{dt} + \lambda T &= 0.
\end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$\begin{aligned}
X(x) &= \begin{cases} C_1 e^{\sqrt{-(\lambda+h)}x} + C_2 e^{-\sqrt{-(\lambda+h)}x} & \text{if } \lambda + h < 0, \\ C_1 x + C_2 & \text{if } \lambda + h = 0, \\ C_1\cos(\sqrt{\lambda + h}x) + C_2\sin(\sqrt{\lambda + h}x) & \text{if } \lambda + h > 0, \end{cases} \\
T(t) &= De^{-(\lambda+h)t}
\end{aligned}$$

where C_1, C_2, D are constants. Now, the boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

are equivalent to

$$
\begin{aligned}
X(0)T(t) &= 0, \\
X(\pi)T(t) &= 0,
\end{aligned}
$$

which imply either $T(t) = 0$ or $X(0) = X(\pi) = 0$. If $T(t) = 0$, then we would have

$$
\begin{aligned}
u(x,t) &= X(x)T(t) \\
&= X(x)0 \\
&= 0,
\end{aligned}
$$

which would be a trivial solution. So we should assume

$$X(0) = X(\pi) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda + h < 0$. Then we have

$$
\begin{aligned}
X(x) &= C_1 e^{\sqrt{-(\lambda+h)}x} + C_2 e^{-\sqrt{-(\lambda+h)}x}, \\
X(0) &= 0,
\end{aligned}
$$

  which implies $C_1 + C_2 = 0$, or $C_2 = -C_1$. So we have

$$
\begin{aligned}
X(x) &= C_1 e^{\sqrt{-(\lambda+h)}x} + C_2 e^{-\sqrt{-(\lambda+h)}x} \\
&= C_1 e^{\sqrt{-(\lambda+h)}x} - C_1 e^{-\sqrt{-(\lambda+h)}x} \\
&= C_1 (e^{\sqrt{-(\lambda+h)}x} - e^{-\sqrt{-(\lambda+h)}x}).
\end{aligned}
$$

  We notice $e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L} \neq 0$ unless $\lambda = 0$. This means

$$
\begin{aligned}
X(x) &= C_1 (e^{\sqrt{-(\lambda+h)}x} + e^{-\sqrt{-(\lambda+h)}x}), \\
X(L) &= 0
\end{aligned}
$$

  implies $C_1 = 0$, and so we have

$$
\begin{aligned}
X(x) &= C_1 (e^{\sqrt{-(\lambda+h)}x} - e^{-\sqrt{-(\lambda+h)}x}) \\
&= 0(e^{\sqrt{-(\lambda+h)}x} + e^{-\sqrt{-(\lambda+h)}x}) \\
&= 0,
\end{aligned}
$$

  which would mean u is a trivial solution. Therefore, the problem has no negative eigenvalues.

- Case 2: Suppose $\lambda + h = 0$. Then we have

$$
\begin{aligned}
X(x) &= C_1 x + C_2, \\
X(0) &= 0,
\end{aligned}
$$

  which implies $C_2 = 0$, and so we have

$$
\begin{aligned}
X(x) &= C_1 x + C_2 \\
&= C_1 x + 0 \\
&= C_1 x.
\end{aligned}
$$

  And

$$
\begin{aligned}
X(x) &= C_1 x, \\
X(\pi) &= 0,
\end{aligned}
$$

  implies $C_1 = 0$. So we have

$$
\begin{aligned}
X(x) &= C_1 x \\
&= 0x \\
&= 0,
\end{aligned}
$$

  which would mean u is a trivial solution. Therefore, 0 is not an eigenvalue of the problem.

- Case 3: Suppose $\lambda + h > 0$. Then we have

$$X(x) = C_1 \cos(\sqrt{\lambda + h}x) + C_2 \sin(\sqrt{\lambda + h}x),$$
$$X(0) = 0,$$

  which implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda + h}x) + C_2 \sin(\sqrt{\lambda + h}x) \\ &= 0 \cos(\sqrt{\lambda + h}x) + C_2 \sin(\sqrt{\lambda + h}x) \\ &= C_2 \sin(\sqrt{\lambda + h}x). \end{aligned}$$

  Next,

$$X(x) = C_2 \sin(\sqrt{\lambda + h}x),$$
$$X(\pi) = 0$$

  implies either $C_2 = 0$ or $\sin(\sqrt{\lambda + h}\pi) = 0$. But $C_2 = 0$ (with $C_1 = 0$) implies $X(x) = 0$ and that $u(x, t)$ would be a trivial solution. So we should assume $\sqrt{\lambda + h}\pi = n\pi$, or equivalently the eigenvalues

$$\lambda_n = \lambda = n^2 - h,$$

  with the corresponding eigenfunctions

$$\begin{aligned} X_n(x) &= C_{2,n} \sin(\sqrt{\lambda_n + h}x) \\ &= C_{2,n} \sin(\sqrt{(n^2 - h) + h}x) \\ &= C_{2,n} \sin(nx), \end{aligned}$$

  as desired.

From the three cases above, we see that the problem only has positive eigenvalues $\lambda_n = n^2 - h$ and their corresponding eigenfunctions $X_n(x) = C_{2,n} \sin(nx)$. We will now use the method of eigenfunction expansion. We can represent, for any fixed t, our solution as

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin(nx),$$

where $T_n(t)$ for $n = 1, 2, 3, \ldots$ are the time-dependent Fourier coefficients. (Note that, unlike Section 5.4 in the textbook, we do not have the term $\frac{1}{2}T_0(t)$ in our representation of $u(x, t)$ because 0 is not an eigenvalue of this problem.) Our derivatives are

$$\begin{aligned} u_t(x, t) &= \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} T_n(t) \sin(nx)\right) \\ &= \sum_{n=1}^{\infty} T_n'(t) \sin(nx) \end{aligned}$$

and

$$\begin{aligned} u_{xx}(x, t) &= \frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} T_n(t) \sin(nx)\right) \\ &= \sum_{n=1}^{\infty} T_n(t) \frac{\partial^2}{\partial x^2}(\sin(nx)) \\ &= \sum_{n=1}^{\infty} -n^2 T_n(t) \sin(nx). \end{aligned}$$

So the partial differential equation

$$u_t - u_{xx} - hu = 0$$

becomes

$$\sum_{n=1}^{\infty} T_n'(t) \sin(nx) - \sum_{n=1}^{\infty} -n^2 T_n(t) \sin(nx) - h \sum_{n=1}^{\infty} T_n(t) \sin(nx) = 0,$$

or equivalently

$$\sum_{n=1}^{\infty} (T_n'(t) + (n^2 - h)T_n(t)) \sin(nx) = 0.$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to obtain the ordinary differential equation

$$T_n'(t) + (n^2 - h)T_n(t) = 0,$$

whose general solution is

$$T_n(t) = B_n e^{-(n^2-h)t},$$

where

$$\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\left(-\frac{1}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Bigg|_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left(-\frac{1}{n}(\pi - x)\cos(nx) - \frac{1}{n^2}\sin(nx)\right)\Bigg|_{\frac{\pi}{2}}^{\pi} \\
&= \frac{4}{\pi^2}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

for $n = 1, 2, 3, \ldots$ are the Fourier coefficients. Therefore, our solution is

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} T_n(t)\sin(nx) \\
&= \sum_{n=1}^{\infty} B_n e^{-(n^2-h)t}\sin(nx) \\
&= \boxed{\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)e^{-(n^2-h)t}\sin(nx)},
\end{aligned}$$

as desired. □

Alternate solution. We employ the substitution $v(x,t) = e^{-ht}u(x,t)$. We obtain the derivatives

$$\begin{aligned}
v_t(x,t) &= -he^{-ht}u(x,t) + e^{-ht}u_t(x,t), \\
v_{xx}(x,t) &= e^{-ht}u_{xx}(x,t).
\end{aligned}$$

So we obtain the partial differential equation

$$\begin{aligned}
v_t - v_{xx} &= (-he^{-ht}u + e^{-ht}u_t) - e^{-ht}u_{xx} \\
&= e^{-ht}(u_t - u_{xx} - hu) \\
&= e^{-ht}\cdot 0 \\
&= 0.
\end{aligned}$$

We also have

$$\begin{aligned}
v(0,t) &= e^{-ht}u(0,t) = e^{-ht}\cdot 0 = 0, \\
v(\pi,t) &= e^{-ht}u(\pi,t) = e^{-ht}\cdot 0 = 0, \\
v(x,0) &= e^{-h\cdot 0}u(x,0) = u(x,0) = x(\pi - x).
\end{aligned}$$

So we have transformed the original problem into a simpler problem:

$$\begin{aligned}
v_t - v_{xx} &= 0 && \text{if } 0 < x < \pi, t > 0, \\
v(0,t) = v(\pi,t) &= 0 && \text{if } t \geq 0, \\
v(x,0) &= x(\pi - x) && \text{if } 0 \leq x \leq \pi.
\end{aligned}$$

According to our solution to Exercise 5.8, part (a), our only eigenvalues are

$$\lambda_n = \lambda = n^2,$$

with the corresponding eigenfunctions

$$\begin{aligned}
X_n(x) &= C_{2,n}\sin(\sqrt{\lambda_n}x) \\
&= C_{2,n}\sin(\sqrt{n^2}x) \\
&= C_{2,n}\sin(nx).
\end{aligned}$$

We will now use the method of eigenfunction expansion. We can represent, for any fixed t,

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(nx),$$

where $T_n(t)$ for $n = 1, 2, 3, \ldots$ are the time-dependent Fourier coefficients. (Note that, unlike Section 5.4 in the textbook, we do not have the term $\frac{1}{2}T_0(t)$ in our representation of $u(x,t)$ because 0 is not an eigenvalue of this problem.) Our derivatives are

$$\begin{aligned} v_t(x,t) &= \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} T_n(t)\sin(nx)\right) \\ &= \sum_{n=1}^{\infty} T_n'(t)\sin(nx) \end{aligned}$$

and

$$\begin{aligned} v_{xx}(x,t) &= \frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} T_n(t)\sin(nx)\right) \\ &= \sum_{n=1}^{\infty} T_n(t)\frac{\partial^2}{\partial x^2}(\sin(nx)) \\ &= \sum_{n=1}^{\infty} -n^2 T_n(t)\sin(nx). \end{aligned}$$

So the partial differential equation

$$v_t - v_{xx} = 0$$

becomes

$$\sum_{n=1}^{\infty} T_n'(t)\sin(nx) - \sum_{n=1}^{\infty} -n^2 T_n(t)\sin(nx) = 0,$$

or equivalently

$$\sum_{n=1}^{\infty} (T_n'(t) + n^2 T_n(t))\sin(nx) = 0.$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to obtain the ordinary differential equation

$$T_n'(t) + n^2 T_n(t) = 0,$$

whose general solution is

$$T_n(t) = B_n e^{-n^2 t},$$

where

$$\begin{aligned} B_n &= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\left(-\frac{1}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Bigg|_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left(-\frac{1}{n}(\pi - x)\cos(nx) - \frac{1}{n^2}\sin(nx)\right)\Bigg|_{\frac{\pi}{2}}^{\pi} \\ &= \frac{4}{\pi^2}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

for $n = 1, 2, 3, \ldots$ are the Fourier coefficients. Therefore, we have

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} T_n(t)\sin(nx) \\ &= \sum_{n=1}^{\infty} B_n e^{-n^2 t}\sin(nx) \\ &= \frac{1}{4\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) e^{-n^2 t}\sin(nx), \end{aligned}$$

and so our solution is

$$\begin{aligned} u(x,t) &= e^{ht}e^{-ht}u(x,t) \\ &= e^{ht}v(x,t) \\ &= e^{ht}\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)e^{-n^2t}\sin(nx) \\ &= \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)e^{-n^2t}e^{ht}\sin(nx) \\ &= \boxed{\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)e^{-(n^2-h)t}\sin(nx)}, \end{aligned}$$

as desired. □

(b) Does $\lim_{t\to\infty} u(x,t)$ exist for all $0 < x < \pi$?
Hint: Distinguish between the following cases: $h < 1$, $h = 1$, $h > 1$.

Solution. Let us look at

$$u(x,t) = \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)e^{-n^2t}e^{ht}\sin(nx),$$

which should be the easiest expression for us to compute the limit. Notice that we have

$$\lim_{t\to\infty} e^{ht} = \begin{cases} 0 & \text{if } h < 1, \\ 1 & \text{if } h = 1, \\ \infty & \text{if } h > 1. \end{cases}$$

(Note that, for the cases $h < 1$ and $h = 1$, $\lim_{t\to\infty} e^{ht}$ converges *uniformly* to the limits 0 and 1, respectively. The uniform convergence will allow us to pass the limit notation inside the summation sign.) If $h < 1$, we have

$$\begin{aligned} \lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)e^{-n^2t}e^{ht}\sin(nx) \\ &= \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\lim_{t\to\infty}e^{-n^2t}e^{ht}\sin(nx) \\ &= \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\cdot 0\cdot\sin(nx) \\ &= 0. \end{aligned}$$

If $h = 1$, we have

$$\begin{aligned} \lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)e^{-n^2t}e^{ht}\sin(nx) \\ &= \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\lim_{t\to\infty}e^{-n^2t}e^{ht}\sin(nx) \\ &= \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\cdot 1\cdot\sin(nx) \\ &= \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin(nx) \\ &= \frac{1}{4\pi}\sin(nx). \end{aligned}$$

If $h > 1$, then we have

$$\begin{aligned} \lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)e^{-n^2t}e^{ht}\sin(nx) \\ &= \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\lim_{t\to\infty}e^{-n^2t}e^{ht}\sin(nx) \\ &= \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\cdot\infty\cdot\sin(nx) \\ &= \infty, \end{aligned}$$

meaning that $u(x,t)$ diverges in t. □

5.10. Consider the problem

$$
\begin{aligned}
u_t &= u_{xx} + \alpha u && 0 < x < 1, t > 0,\\
u(0,t) &= u(1,t) = 0 && t \ge 0,\\
u(x,0) &= f(x) && 0 \le x \le 1
\end{aligned}
$$

for any $f \in C([0,1])$.

(a) Assume $\alpha = -1$ and $f(x) = x$. Solve the problem.

Remark. *One can solve Exercise 5.10, part (a) in multiple ways, especially when the the problem statement of this textbook exercise does not specify which method to use. Specifically here, one can employ the usual method of separation of variables, the method of eigenfunction expansion (see my solution of Exercise 5.9, part (a)), or an exponential solution in tandem with the method of eigenfunction expansion (see my alternate solution of Exercise 5.9, part (a)). As requested by the students in my T.A. office hours, I will employ the method of separation of variables for this exercise.*

Solution. The given problem with $\alpha = -1$ and $f(x) = x$ becomes

$$
\begin{aligned}
u_t &= u_{xx} - u && 0 < x < 1, t > 0,\\
u(0,t) &= u(1,t) = 0 && t \ge 0,\\
u(x,0) &= x && 0 \le x \le 1
\end{aligned}
$$

Following the method of separation of variables, we want to find a solution of the form

$$u(x,t) = X(x)T(t).$$

Our partial derivatives are

$$
\begin{aligned}
u_t(x,t) &= X(x)T_t(t),\\
u_{xx}(x,t) &= X_{xx}(x)T(t).
\end{aligned}
$$

So the partial differential equation

$$u_t = u_{xx} - u$$

becomes

$$X(x)T_t(t) = X_{xx}(x)T(t) - X(x)T(t),$$

which we can algebraically rearrange to write

$$\frac{X_{xx}(x) - X(x)}{X(x)} = \frac{T_t(t)}{T(t)} = -\lambda,$$

where λ is a constant in both x and t. This produces the system of two ordinary differential equations

$$
\begin{aligned}
\frac{d^2X}{dx^2} + (\lambda - 1)X &= 0\\
\frac{dT}{dt} + \lambda kT &= 0.
\end{aligned}
$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$
\begin{aligned}
X(x) &= \begin{cases} C_1 e^{\sqrt{-(\lambda-1)}x} + C_2 e^{-\sqrt{-(\lambda-1)}x} & \text{if } \lambda - 1 < 0,\\ C_1 x + C_2 & \text{if } \lambda - 1 = 0,\\ C_1\cos(\sqrt{\lambda - 1}x) + C_2\sin(\sqrt{\lambda - 1}x) & \text{if } \lambda - 1 > 0,\end{cases}\\
T(t) &= De^{-(\lambda-1)t},
\end{aligned}
$$

where C_1, C_2, D are constants. Now, the boundary conditions

$$u_x(0,t) = u_x(L,t) = 0$$

are equivalent to

$$
\begin{aligned}
\frac{d}{dx}X(0)T(t) &= 0,\\
\frac{d}{dx}X(L)T(t) &= 0,
\end{aligned}
$$

which imply either $T(t) = 0$ or $\frac{d}{dx}X(0) = \frac{d}{dx}X(L) = 0$. If $T(t) = 0$, then we would have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= X(x)0 \\ &= 0, \end{aligned}$$

which would be a trivial solution. But we are really interested in finding a nontrivial solution. So we should assume

$$\frac{d}{dx}X(0) = \frac{d}{dx}X(L) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda - 1 < 0$. Then we have

  $$\begin{aligned} X(x) &= C_1 e^{\sqrt{-(\lambda-1)}x} + C_2 e^{-\sqrt{-(\lambda-1)}x}, \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

  which implies

  $$\begin{aligned} \frac{d}{dx}X(x) &= \sqrt{-(\lambda-1)}(C_1 e^{\sqrt{-(\lambda-1)}x} - C_2 e^{-\sqrt{-(\lambda-1)}x}), \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

  which implies $\sqrt{-(\lambda-1)}(C_1 - C_2) = 0$. As we assumed $\lambda - 1 < 0$ in this case, we have $\sqrt{-(\lambda-1)} \neq 0$, and so we conclude $C_1 - C_2 = 0$, or equivalently $C_1 = C_2$. So we have

  $$\begin{aligned} X(x) &= C_1 e^{\sqrt{-(\lambda-1)}x} + C_2 e^{-\sqrt{-(\lambda-1)}x} \\ &= C_1 e^{\sqrt{-(\lambda-1)}x} + C_1 e^{-\sqrt{-(\lambda-1)}x} \\ &= C_1(e^{\sqrt{-(\lambda-1)}x} + e^{-\sqrt{-(\lambda-1)}x}). \end{aligned}$$

  We notice $e^{\sqrt{-(\lambda-1)}L} + e^{-\sqrt{-(\lambda-1)}L} \neq 0$. This means

  $$\begin{aligned} X(x) &= C_1(e^{\sqrt{-(\lambda-1)}x} + e^{-\sqrt{-(\lambda-1)}x}), \\ X(L) &= 0 \end{aligned}$$

  implies $C_1 = 0$, and so we have

  $$\begin{aligned} X(x) &= C_1(e^{\sqrt{-(\lambda-1)}x} + e^{-\sqrt{-(\lambda-1)}x}) \\ &= 0(e^{\sqrt{-(\lambda-1)}x} + e^{-\sqrt{-(\lambda-1)}x}) \\ &= 0. \end{aligned}$$

  Therefore, we have

  $$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= 0T(t) \\ &= 0, \end{aligned}$$

  which is a trivial solution.
- Case 2: Suppose $\lambda - 1 = 0$. Then we have

  $$\begin{aligned} X(x) &= C_1 x + C_2, \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

  which implies

  $$\begin{aligned} \frac{d}{dx}X(x) &= C_1, \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

which implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 x + C_2 \\ &= 0x + C_2 \\ &= C_2, \end{aligned}$$

which already satisfies $\frac{d}{dx}X(L) = 0$. Therefore, if we write $\frac{A_0}{2} = C_2 D$, then we have

$$\begin{aligned} u_0(x,t) &= X(x)T(t) \\ &= C_2 D e^{-(\lambda-1)k(0)} \\ &= C_2 D \\ &= \frac{A_0}{2}, \end{aligned}$$

which is a nontrivial solution.

- Case 3: Suppose $\lambda - 1 > 0$. Then we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda-1}x) + C_2 \sin(\sqrt{\lambda-1}x), \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dx}X(x) &= \sqrt{\lambda-1}(-C_1 \sin(\sqrt{\lambda-1}x) + C_2 \cos(\sqrt{\lambda-1}x)), \\ \frac{d}{dx}X(0) &= 0, \end{aligned}$$

which implies $\sqrt{\lambda-1}C_2 = 0$, which in turn implies either $\sqrt{\lambda-1} = 0$ or $C_2 = 0$. But $\sqrt{\lambda-1} = 0$ implies $\lambda - 1 = 0$, which contradicts our assumption $\lambda - 1 > 0$ for this case. So we must have $C_2 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda-1}x) + C_2 \sin(\sqrt{\lambda-1}x) \\ &= C_1 \cos(\sqrt{\lambda-1}x) + 0 \sin(\sqrt{\lambda-1}x) \\ &= C_1 \cos(\sqrt{\lambda-1}x) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dx}X(x) &= \sqrt{\lambda-1}(-C_1 \sin(\sqrt{\lambda-1}x) + C_2 \cos(\sqrt{\lambda-1}x)) \\ &= \sqrt{\lambda-1}(-C_1 \sin(\sqrt{\lambda-1}x) + 0 \cos(\sqrt{\lambda-1}x)) \\ &= -\sqrt{\lambda-1}C_1 \sin(\sqrt{\lambda-1}x), \end{aligned}$$

Next,

$$\begin{aligned} \frac{d}{dx}X(x) &= -\sqrt{\lambda-1}C_1 \sin(\sqrt{\lambda-1}x), \\ \frac{d}{dx}X(L) &= 0 \end{aligned}$$

implies $\sqrt{\lambda-1}C_1 \sin(\sqrt{\lambda-1}L) = 0$. As we assumed $\lambda - 1 > 0$ in this case, we have $\sqrt{\lambda-1} \neq 0$, and so we conclude $C_1 \sin(\sqrt{\lambda-1}L) = 0$. If $C_1 = 0$, then we would have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda-1}x) + C_2 \sin(\sqrt{\lambda-1}x) \\ &= 0 \cos(\sqrt{\lambda-1}x) + 0 \sin(\sqrt{\lambda-1}x) \\ &= 0, \end{aligned}$$

which would imply

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= 0T(t) \\ &= 0, \end{aligned}$$

meaning that $u(x,t)$ is a trivial solution. So we should assume $\sin(\sqrt{\lambda-1}L) = 0$, which implies $\sqrt{\lambda-1}L = n\pi$, or equivalently

$$\lambda_n = \lambda = \left(\frac{n\pi}{L}\right)^2 + 1,$$

and so we have

$$\begin{aligned} X_n(x) &= C_{1,n}\cos(\sqrt{\lambda_n - 1}x) \\ &= C_{1,n}\cos(\sqrt{((n\pi)^2+1)-1}x) \\ &= C_{1,n}\cos(n\pi x) \end{aligned}$$

and

$$\begin{aligned} T_n(t) &= D_n e^{-\lambda_n kt} \\ &= D_n e^{-(\frac{n\pi}{L})^2 kt} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Therefore, if we write $A_n = C_{1,n}D_n$, then we have

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ &= (C_{1,n}\cos(n\pi x))(D_n e^{-(n\pi)^2 kt}) \\ &= C_{1,n}D_n e^{-(n\pi)^2 kt}\cos(n\pi x) \\ &= A_n e^{-(n\pi)^2 kt}\cos(n\pi x) \end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial solution, as desired.

We recall that an addition of solutions is again a solution. So that means, as we have established already that each $u_n(x,t)$ is a nontrivial solution for $n = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned} u(x,t) &= u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) \\ &= \boxed{\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 kt}\cos(n\pi x)}, \end{aligned}$$

where

$$\begin{aligned} A_0 &= \frac{2}{1}\int_0^1 f(x)\,dx \\ &= 2\int_0^1 x\,dx \\ &= 2\frac{x^2}{2}\bigg|_0^1 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} A_n &= \frac{2}{1}\int_0^1 f(x)\cos(n\pi x)\,dx \\ &= 2\int_0^1 x\cos(n\pi x)\,dx \\ &= \frac{1}{n\pi}x\sin(n\pi x)\bigg|_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\,dx \\ &= 0 - \frac{1}{(n\pi)^2}\cos(n\pi x)\bigg|_0^1 \\ &= \cos(n\pi) - \cos(0) \\ &= (-1)^n - 1 \end{aligned}$$

for $n = 1, 2, 3, \ldots$ are the Fourier coefficients, is also a solution of the problem. □

5.15. Using the energy method, prove the uniqueness for the problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= F(x,t) && \text{if } 0 < x < L, t > 0, \\ u_x(0,t) = t^2, u(L,t) &= -t && \text{if } t \geq 0, \\ u(x,0) &= x^2 - L^2 && \text{if } 0 \leq x \leq L, \\ u_t(x,0) &= \sin^2\left(\frac{\pi x}{L}\right) && \text{if } 0 \leq x \leq L. \end{aligned}$$

Proof. Let u_1 and u_2 be two solutions of the problem, and define $w = u_1 - u_2$. Then we have the partial differential equation

$$\begin{aligned} w_{tt} - c^2 u_{xx} &= (u_{1,tt} - u_{2,tt}) - c^2((u_1)_{xx} - (u_2)_{xx}) \\ &= u_{1,tt} - c^2 y_{1,xx} - (u_{2,tt} - c^2 u_{xx}) \\ &= F(x,t) - F(x,t) \\ &= 0. \end{aligned}$$

We also have the boundary conditions

$$\begin{aligned} w_x(0,t) &= (u_1)_x(0,t) - (u_2)_x(0,t) \\ &= t^2 - t^2 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} w(L,t) &= u_1(L,t) - u_2(L,t) \\ &= -t - (-t) \\ &= 0. \end{aligned}$$

We also have the initial conditions

$$\begin{aligned} w(x,0) &= u_1(x,0) - u_2(x,0) \\ &= (x^2 - L^2) - (x^2 - L^2) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} w_t(x,0) &= u_{1,t}(x,0) - u_{2,t}(x,0) \\ &= \sin^2\left(\frac{\pi x}{L}\right) - \sin^2\left(\frac{\pi x}{L}\right) \\ &= 0. \end{aligned}$$

So we have transformed the original problem into

$$\begin{aligned} w_{tt} - c^2 w_{xx} &= 0 \quad \text{if } 0 < x < L, t > 0, \\ w_x(0,t) = w(L,t) &= 0 \quad \text{if } t \geq 0, \\ w(x,0) = w_t(x,0) &= 0 \quad \text{if } 0 \leq x \leq L. \end{aligned}$$

Now, we will employ the energy method to establish $w = 0$. Multiply both sides of the partial differential equation

$$w_{tt} - c^2 w_{xx} = 0$$

by w_t to write

$$w_t(w_{tt} - c^2 w_{xx}) = w_t 0,$$

or equivalently

$$w_t w_{tt} - c^2 w_t w_{xx} = 0.$$

In fact, we can rewrite the left-hand side as

$$\frac{1}{2}\frac{d}{dt}(w_t)^2 - c^2 \frac{d}{dt}(w_x)^2 = 0,$$

or equivalently

$$\frac{d}{dt}\left(\frac{1}{2}((w_t)^2 - c^2(w_x)^2)\right) = 0.$$

Integrate over the domain $0 < x < L$ both sides to write

$$\int_0^L \frac{d}{dt}\left(\frac{1}{2}(w_t)^2 - c^2(w_x)^2\right) dx = \int_0^L 0\, dx,$$

or equivalently

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^L (w_t)^2 - c^2(w_x)^2\, dx\right) = 0.$$

This motivates us to define the energy

$$E(t) = \frac{1}{2}\int_0^L (w_t)^2 - c^2(w_x)^2\, dx.$$

Then we have

$$\begin{aligned}\frac{d}{dt}E(t) &= \frac{d}{dt}\left(\frac{1}{2}\int_0^L (w_t)^2 - c^2(w_x)^2\,dx\right)\\ &= 0,\end{aligned}$$

meaning that $E(t)$ is constant in t. But as we have

$$\begin{aligned}E(0) &= \frac{1}{2}\int_0^L (w_t(x,0))^2 - c^2(w_x(x,0))^2\,dx\\ &= \frac{1}{2}\int_0^L 0^2 - c^2 0^2\,dx\\ &= 0,\end{aligned}$$

we conclude $E(t) = 0$ for all $t \geq 0$. So we have $w_t(x,0) = 0$ and $w_x(x,0) = 0$. Now, $w_t(x,0) = 0$ implies $w(x,t) = C(x)$ and $w_x(x,0) = 0$ implies $w(x,t) = D(t)$. But the only possibility that $w(x,t) = C(x) = D(t)$ holds is $w(x,t) = C$, where C is a constant. But $w(x,0) = 0$ implies $C = 0$, and so $w(x,t) = 0$. Finally, $w = 0$ implies $u_1 - u_2 = 0$, or $u_1 = u_2$, meaning that the solution of the original problem is unique. □