MATH 146C discussion | Ryan Ta
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Solutions to suggested homework problems from
An Introduction to Partial Differential Equations by Yehuda Pinchover and Jacob Rubinstein

Suggested problems: Exercises 7.2, 7.3, 7.4, 7.7, 7.8, 7.11, 7.14(a), 7.15, 7.20, 7.22

Note: Almost all steps for solving an ordinary differential equation (for example, any material from MATH 046 at UC Riverside) are omitted from my solutions for purposes of brevity.

7.2. Prove uniqueness for the Dirichlet and Neumann problems for the reduced Helmholtz equation

$$u - ku = 0$$

in a bounded planar domain $D \subset \mathbb{R}^2$, where k is a positive constant.

Solution. Consider the Dirichlet problem

$$\begin{aligned} \Delta u - ku &= 0 \qquad (x,y) \in D, \\ u(x,y) &= g(x,y) \quad (x,y) \in \partial D. \end{aligned}$$

Let u_1 and u_2 be two solutions of the Dirichlet problem, and let $w_1 := u_1 - u_2$. Then w_1 solves

$$\begin{aligned} \Delta w_1 - kw_1 &= 0 \quad (x,y) \in D, \\ w_1(x,y) &= 0 \quad (x,y) \in \partial D. \end{aligned}$$

Now, we recall Green's third identity (also known as integration by parts),

$$\iint_D \nabla u \cdot \nabla v \, dx \, dy = \int_{\partial D} u \frac{\partial u}{\partial n} \, ds - \iint_D u \Delta v \, dx \, dy.$$

Substitute $u = w_1$ and $v = w_1$ into Green's third identity in order to obtain

$$\iint_D \nabla w_1 \cdot \nabla w_1 \, dx \, dy = \int_{\partial D} w_1 \frac{\partial w_1}{\partial n} \, ds - \iint_D w_1 \Delta w_1 \, dx \, dy.$$

The definition of the dot product of two vectors implies in particular $\nabla w_1 \cdot \nabla w_1 = |\nabla w_1|^2$. And the partial differential equation $\Delta w_1 - kw_1 = 0$ is of course equivalent to $\Delta w_1 = kw_1$. So we obtain, in fact,

$$\iint_D |\nabla w_1|^2 \, dx \, dy = \int_{\partial D} w_1 \frac{\partial w_1}{\partial n} \, ds - k \iint_D (w_1)^2 \, dx \, dy.$$

Finally, as we have $w_1 = 0$ on ∂D, we obtain

$$\iint_D |\nabla w_1|^2 \, dx \, dy = \int_{\partial D} 0 \frac{\partial w_1}{\partial n} \, ds - k \iint_D (w_1)^2 \, dx \, dy,$$

or more succinctly

$$\iint_D |\nabla w_1|^2 \, dx \, dy = -k \iint_D (w_1)^2 \, dx \, dy$$

Observe that, for any $k > 0$, the left hand side and right hand side satisfy

$$\begin{aligned} \iint_D |\nabla w_1|^2 \, dx \, dy &\geq 0, \\ -k \iint_D (w_1)^2 \, dx \, dy &\leq 0, \end{aligned}$$

respectively. The only way these inequalities hold true simultaneously is only when they satisfy

$$\begin{aligned} \iint_D |\nabla w_1|^2 \, dx \, dy &= 0, \\ -k \iint_D (w_1)^2 \, dx \, dy &= 0. \end{aligned}$$

In particular, as $k > 0$, the equality

$$-k \iint_D (w_1)^2 \, dx \, dy = 0$$

implies $(w_1)^2 = 0$, which in turn gives $w_1 = 0$, or equivalently $u_1 = u_2$. This establishes the uniqueness of the Dirichlet problem. Next, consider the Neumann problem

$$\begin{aligned} \Delta u - ku &= 0 \qquad (x,y) \in D, \\ \partial_n u(x,y) &= g(x,y) \quad (x,y) \in \partial D. \end{aligned}$$

Let u_3 and u_4 be two solutions of the Dirichlet problem, and let $w_2 := u_3 - u_4$. Then w_2 solves

$$\begin{aligned} \Delta w_2 - kw_2 &= 0 \quad (x,y) \in D, \\ \partial_n w_2(x,y) &= 0 \quad (x,y) \in \partial D. \end{aligned}$$

Now, we will establish $w_2 = 0$ in D. As established previously for w_1, we obtain

$$\iint_D |\nabla w_2|^2 \, dx\, dy = \int_{\partial D} w_2 \frac{\partial w_2}{\partial n} \, ds - k \iint_D (w_2)^2 \, dx\, dy.$$

Finally, as we have $\frac{\partial w_2}{\partial n} = 0$ on ∂D, we obtain

$$\iint_D |\nabla w_2|^2 \, dx\, dy = \int_{\partial D} w_2 0 \, ds - k \iint_D (w_2)^2 \, dx\, dy.$$

or more succinctly

$$\iint_D |\nabla w_2|^2 \, dx\, dy = -k \iint_D (w_2)^2 \, dx\, dy$$

Following the rest of our proof for w_1, we conclude $w_2 = 0$, or equivalently $u_3 = u_4$, in D. This establishes the uniqueness of the Neumann problem. □

7.3. Solve the problem

$$\begin{aligned} \Delta u + ku &= 0 \quad 0 < x < \pi, 0 < y < \pi, \\ u(0,y) &= 1 \quad 0 < y < \pi, \\ u(\pi,y) = u(x,0) = u(x,\pi) &= 0 \quad 0 < x < \pi. \end{aligned}$$

Solution. We want to find a solution of the form

$$u(x,y) = X(x)Y(y).$$

Our partial derivatives are

$$\begin{aligned} u_{xx}(x,t) &= X_{xx}(x)Y(y), \\ u_{yy}(x,t) &= X(x)Y_{yy}(y) \end{aligned}$$

So the partial differential equation

$$u_{xx} + u_{yy} - ku = \Delta u - ku = 0$$

becomes

$$X_{xx}(x)Y(y) + X(x)Y_{yy}(y) - kX(x)Y(y) = 0,$$

which we can algebraically rearrange to write

$$-\frac{X_{xx}(x) - kX(x)}{X(x)} = \frac{Y_{yy}(y)}{Y(y)} = -\lambda,$$

where λ is a constant in both x and y. This produces the system of two ordinary differential equations

$$\begin{aligned} \frac{d^2X}{dx^2} - (\lambda + k)X &= 0 \\ \frac{d^2Y}{dy^2} + \lambda Y &= 0. \end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$X(x) = \begin{cases} C_1 \cos(\sqrt{-(\lambda+k)}x) + C_2 \sin(\sqrt{-(\lambda+k)}x) & \text{if } \lambda + k < 0, \\ C_1 x + C_2 & \text{if } \lambda + k = 0, \\ C_1 e^{\sqrt{\lambda+k}x} + C_2 e^{-\sqrt{\lambda+k}x} & \text{if } \lambda + k > 0, \end{cases}$$

$$Y(y) = \begin{cases} D_1 e^{\sqrt{-\lambda}y} + D_2 e^{-\sqrt{-\lambda}y} & \text{if } \lambda < 0, \\ D_1 y + D_2 & \text{if } \lambda = 0, \\ D_1 \cos(\sqrt{\lambda}y) + D_2 \sin(\sqrt{\lambda}y) & \text{if } \lambda > 0, \end{cases}$$

where C_1, C_2, D_1, D_2 are constants. Now, the boundary conditions

$$u(\pi, y) = u(x, 0) = u(x, \pi) = 0$$

are equivalent to

$$X(\pi)Y(y) = X(x)Y(0) = X(x)Y(\pi) = 0,$$

which imply either $X(x) = Y(y) = 0$ or $X(\pi) = Y(0) = Y(\pi) = 0$. If we assume either $X(x) = 0$ or $Y(y) = 0$, then in either case we would have a trivial solution. But we are really interested in finding a nontrivial solution. So we should assume

$$X(\pi) = Y(0) = Y(\pi) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then

  $$\begin{aligned} &Y(y) = D_1 e^{\sqrt{-\lambda} y} + D_2 e^{-\sqrt{-\lambda} y}, \\ &Y(0) = 0 \end{aligned}$$

  implies $D_2 = -D_1$, and so we have

  $$\begin{aligned} Y(y) &= D_1 e^{\sqrt{-\lambda} y} + D_2 e^{-\sqrt{-\lambda} y} \\ &= D_1 e^{\sqrt{-\lambda} y} - D_1 e^{-\sqrt{-\lambda} y} \\ &= D_1(e^{\sqrt{-\lambda} y} - e^{-\sqrt{-\lambda} y}). \end{aligned}$$

  Now, if $\lambda < 0$, then $e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi} \neq 0$. This means

  $$\begin{aligned} &Y(y) = D_1(e^{\sqrt{-\lambda} y} - D_2 e^{-\sqrt{-\lambda} y}), \\ &Y(\pi) = 0 \end{aligned}$$

  implies $C_1 = 0$, and so we have

  $$\begin{aligned} X(x) &= D_1(e^{\sqrt{-\lambda} y} - e^{-\sqrt{-\lambda} y}) \\ &= 0(e^{\sqrt{-\lambda} y} - e^{-\sqrt{-\lambda} y}) \\ &= 0. \end{aligned}$$

  Therefore, we have

  $$\begin{aligned} u(x, y) &= X(x)Y(y) \\ &= 0Y(y) \\ &= 0, \end{aligned}$$

  which is a trivial solution.
- Case 2: Suppose $\lambda = 0$. Then

  $$\begin{aligned} &Y(y) = D_1 y + D_2, \\ &Y(0) = 0 \end{aligned}$$

  implies $D_2 = 0$, and so we have

  $$\begin{aligned} Y(y) &= D_1 y + D_2 \\ &= D_1 y + 0 \\ &= D_1 y. \end{aligned}$$

  Next,

  $$\begin{aligned} &Y(y) = D_1 y, \\ &Y(\pi) = 0 \end{aligned}$$

  implies $D_1 = 0$, and so we have

  $$\begin{aligned} Y(y) &= D_1 y \\ &= 0y \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u(x, y) &= X(x)Y(y) \\ &= 0Y(y) \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 3: Suppose $\lambda > 0$. Then

$$\begin{aligned} &Y(y) = D_1 \cos(\sqrt{\lambda} y) + D_2 \sin(\sqrt{\lambda} y), \\ &Y(0) = 0 \end{aligned}$$

implies $D_1 = 0$, and so we have

$$\begin{aligned} Y(y) &= D_1 \cos(\sqrt{\lambda} y) + D_2 \sin(\sqrt{\lambda} y) \\ &= 0 \cos(\sqrt{\lambda} x) + D_2 \sin(\sqrt{\lambda} y) \\ &= D_2 \sin(\sqrt{\lambda} y). \end{aligned}$$

Next,

$$\begin{aligned} &Y(y) = D_2 \sin(\sqrt{\lambda} y), \\ &Y(\pi) = 0 \end{aligned}$$

implies $\sin(\sqrt{\lambda} y) = 0$, which in turn implies $\sqrt{\lambda}\pi = n\pi$, or equivalently

$$\lambda_n = \lambda = n^2,$$

and so we have

$$\begin{aligned} Y_n(y) &= D_{2,n} \sin(\sqrt{\lambda_n} y) \\ &= D_{2,n} \sin(\sqrt{n^2} y) \\ &= D_{2,n} \sin(ny). \end{aligned}$$

Also, as we are in the case of $\lambda > 0$ and we are given $k > 0$ from the problem statement, it follows that we have $\lambda + k > 0$. Therefore, we get

$$\begin{aligned} X_n(x) &= C_{1,n} e^{\sqrt{\lambda_n + k} x} + C_{2,n} e^{-\sqrt{\lambda_n + k} x} \\ &= C_{1,n} e^{\sqrt{n^2 + k} x} + C_{2,n} e^{-\sqrt{n^2 + k} x} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Therefore, if we write $A_n = C_{2,n} D_{1,n}$ and $B_n = C_{2,n} D_{2,n}$, then we have

$$\begin{aligned} u_n(x, y) &= X_n(x) Y_n(y) \\ &= (C_{1,n} e^{\sqrt{n^2+k} x} + C_{2,n} e^{-\sqrt{n^2+k} x})(D_{2,n} \sin(ny)) \\ &= (C_{1,n} D_{2,n} e^{\sqrt{n^2+k} x} + C_{2,n} D_{2,n} e^{-\sqrt{n^2+k} x}) \sin(ny) \\ &= (A_n e^{\sqrt{n^2+k} x} + B_n e^{-\sqrt{n^2+k} x}) \sin(ny) \end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial solution, as desired.

We recall that an addition of solutions is again a solution. So that means, as we have established already that each $u_n(x, t)$ is a nontrivial solution for $n = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} u_n(x, y) \\ &= \sum_{n=1}^{\infty} (A_n e^{\sqrt{n^2+k} x} + B_n e^{-\sqrt{n^2+k} x}) \sin(ny) \end{aligned}$$

is also a nontrivial solution of the problem. Next, we will compute the Fourier coefficients A_n, B_n. We have

$$\begin{aligned} 1 = u(0, y) &= \sum_{n=1}^{\infty} (A_n + B_n) \sin(ny), \\ 0 = u(\pi, y) &= \sum_{n=1}^{\infty} (A_n e^{\sqrt{n^2+k}\pi} + B_n e^{-\sqrt{n^2+k}\pi}) \sin(ny). \end{aligned}$$

Now, recall

$$\int_0^\pi \sin(ny)\sin(my)\,dy = \begin{cases} \frac{\pi}{2} & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Consequently, we obtain

$$\begin{aligned} \int_0^\pi 1\sin(ny)\,dy &= \int_0^\pi \sum_{m=1}^{\infty}(A_m + B_m)\sin(my)\sin(ny)\,dy \\ &= \sum_{m=1}^{\infty}(A_m + B_m)\int_0^\pi \sin\left(\frac{\sqrt{m^2+k\pi}}{b}x\right)\sin\left(\frac{\sqrt{m^2+k\pi}}{b}x\right)\,dx \\ &= (A_n + B_n)\frac{\pi}{2} \end{aligned}$$

and

$$\begin{aligned} \int_0^\pi 0\sin(ny)\,dy &= \int_0^\pi \sum_{m=1}^{\infty}(A_m e^{\sqrt{m^2+k}\pi} + B_m e^{-\sqrt{m^2+k}\pi})\sin(my)\sin(ny)\,dy \\ &= \sum_{m=1}^{\infty}(A_m e^{\sqrt{m^2+k}\pi} + B_m e^{-\sqrt{m^2+k}\pi})\int_0^\pi \sin(my)\sin(ny)\,dy \\ &= (A_n e^{\sqrt{n^2+k}\pi} + B_n e^{-\sqrt{n^2+k}\pi})\frac{\pi}{2}. \end{aligned}$$

In other words, we have the system

$$\begin{aligned} A_n + B_n &= \frac{2}{\pi}\int_0^\pi 1\sin(ny)\,dy = \begin{cases} \frac{4}{\pi n} & \text{if } n = 1,3,5,\ldots \\ 0 & \text{if } n = 2,4,6,\ldots, \end{cases} \\ A_n e^{\sqrt{n^2+k}\pi} + B_n e^{-\sqrt{n^2+k}\pi} &= \frac{2}{\pi}\int_0^\pi 0\sin(ny)\,dy = 0, \end{aligned}$$

which we can solve simultaneously to obtain the coefficients

$$\begin{aligned} A_n &= -\frac{2}{\pi}\frac{e^{-\sqrt{n^2+k}\pi}}{n\sinh(n\pi)}, \\ B_n &= \frac{2}{\pi}\frac{e^{\sqrt{n^2+k}\pi}}{n\sinh(n\pi)}. \end{aligned}$$

So our formal solution is

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty}(A_n e^{\sqrt{n^2+k}x} + B_n e^{-\sqrt{n^2+k}x})\sin(ny) \\ &= \frac{2}{\pi}\sum_{n=1,3,5,\ldots}\frac{1}{n\sinh(n\pi)}(-e^{-\sqrt{n^2+k}\pi}e^{\sqrt{n^2+k}x} + e^{\sqrt{n^2+k}\pi}e^{-\sqrt{n^2+k}x})\sin(ny) \\ &= \frac{2}{\pi}\sum_{n=1,3,5,\ldots}\frac{1}{n\sinh(n\pi)}(e^{\sqrt{n^2+k}(\pi-x)} - e^{-\sqrt{n^2+k}(\pi-x)})\sin(ny) \\ &= \frac{4}{\pi}\sum_{n=1,3,5,\ldots}\frac{1}{n\sinh(n\pi)}\frac{e^{\sqrt{n^2+k}(\pi-x)} - e^{-\sqrt{n^2+k}(\pi-x)}}{2}\sin(ny) \\ &= \frac{4}{\pi}\sum_{n=1,3,5,\ldots}\frac{1}{\sinh(n\pi)}\sinh(\sqrt{n^2+k}(\pi-x))\sin(ny) \\ &= \boxed{\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{\sinh((2n-1)\pi)}\sinh(\sqrt{(2n-1)^2+k}(\pi-x))\sin((2n-1)y)}, \end{aligned}$$

as desired. □

7.4. Solve the problem

$$\begin{aligned} \Delta u = 0 &\quad 0 < x < \pi, 0 < y < \pi, \\ u(x,0) = u(x,\pi) = 1 &\quad 0 \leq x \leq \pi, \\ u(0,y) = u(\pi,y) = 0 &\quad 0 \leq y \leq \pi. \end{aligned}$$

Solution. We want to find a solution of the form

$$u(x, y) = X(x)Y(y).$$

Our partial derivatives are

$$u_{xx}(x,t) = X_{xx}(x)Y(y),$$
$$u_{yy}(x,t) = X(x)Y_{yy}(y)$$

So the partial differential equation

$$u_{xx} + u_{yy} = \Delta u = 0$$

becomes

$$X_{xx}(x)Y(y) + X(x)Y_{yy}(y) = 0,$$

which we can algebraically rearrange to write

$$\frac{X_{xx}(x)}{X(x)} = -\frac{Y_{yy}(y)}{Y(y)} = -\lambda,$$

where λ is a constant in both x and y. This produces the system of two ordinary differential equations

$$\frac{d^2X}{dx^2} + \lambda X = 0$$
$$\frac{d^2Y}{dy^2} - \lambda Y = 0.$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$X(x) = \begin{cases} C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0, \\ C_1 x + C_2 & \text{if } \lambda = 0, \\ C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) & \text{if } \lambda > 0, \end{cases}$$

$$Y(y) = \begin{cases} D_1 \cos(\sqrt{-\lambda}y) + D_2 \sin(\sqrt{-\lambda}y) & \text{if } \lambda < 0, \\ D_1 y + D_2 & \text{if } \lambda = 0, \\ D_1 e^{\sqrt{\lambda}y} + D_2 e^{-\sqrt{-\lambda}y} & \text{if } \lambda > 0, \end{cases}$$

where C_1, C_2, D_1, D_2 are constants. Now, the boundary conditions

$$u(0, y) = u(\pi, y) = 0$$

are equivalent to

$$X(0)Y(y) = 0,$$
$$X(\pi)Y(y) = 0,$$

which imply either $Y(y) = 0$ or $X(0) = X(\pi) = 0$. If $Y(y) = 0$, then we would have

$$\begin{aligned} u(x, y) &= X(x)Y(y) \\ &= X(x)0 \\ &= 0, \end{aligned}$$

which would be a trivial solution. But we are really interested in finding a nontrivial solution. So we should assume

$$X(0) = X(\pi) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then

  $$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x},$$
  $$X(0) = 0$$

  implies $C_2 = -C_1$, and so we have

  $$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \\ &= C_1 e^{\sqrt{-\lambda}x} - C_1 e^{-\sqrt{-\lambda}x} \\ &= C_1 (e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}). \end{aligned}$$

Now, if $\lambda < 0$, then $e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi} \neq 0$. This means

$$\begin{aligned} X(x) &= C_1(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}), \\ X(\pi) &= 0 \end{aligned}$$

implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}) \\ &= 0(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}) \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u(x, y) &= X(x)Y(y) \\ &= 0Y(y) \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 2: Suppose $\lambda = 0$. Then

$$\begin{aligned} X(x) &= C_1x + C_2, \\ X(0) &= 0 \end{aligned}$$

implies $C_2 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1x + C_2 \\ &= C_1x + 0 \\ &= C_1x. \end{aligned}$$

Next,

$$\begin{aligned} X(x) &= C_1x, \\ X(\pi) &= 0 \end{aligned}$$

implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1x \\ &= 0x \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u(x, y) &= X(x)Y(y) \\ &= 0Y(y) \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 3: Suppose $\lambda > 0$. Then

$$\begin{aligned} X(x) &= C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x), \\ X(0) &= 0 \end{aligned}$$

implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x) \\ &= 0\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x) \\ &= C_2\sin(\sqrt{\lambda}x). \end{aligned}$$

Next,

$$\begin{aligned} X(x) &= C_2\sin(\sqrt{\lambda}x), \\ X(\pi) &= 0 \end{aligned}$$

implies $\sin(\sqrt{\lambda}\pi) = 0$, which in turn implies $\sqrt{\lambda}\pi = n\pi$, or equivalently

$$\lambda_n = \lambda = n^2,$$

and so we have

$$\begin{aligned} X_n(x) &= C_{2,n}\sin(\sqrt{\lambda_n}x) \\ &= C_{2,n}\sin(\sqrt{n^2}x) \\ &= C_{2,n}\sin(nx) \end{aligned}$$

and

$$\begin{aligned} Y_n(y) &= D_{1,n}e^{\sqrt{\lambda_n}y} + D_{2,n}e^{-\sqrt{\lambda_n}y} \\ &= D_{1,n}e^{\sqrt{n^2}y} + D_{2,n}e^{-\sqrt{n^2}y} \\ &= D_{1,n}e^{ny} + D_{2,n}e^{-ny} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Therefore, if we write $A_n = C_{2,n}D_{1,n}$ and $B_n = C_{2,n}D_{2,n}$, then we have

$$\begin{aligned} u_n(x,y) &= X_n(x)Y_n(y) \\ &= (C_{2,n}\sin(nx))(D_{1,n}e^{ny} + D_{2,n}e^{-ny}) \\ &= \sin(nx)(C_{2,n}D_{1,n}e^{ny} + C_{2,n}D_{2,n}e^{-ny}) \\ &= \sin(nx)(A_ne^{ny} + B_ne^{-ny}) \end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial solution, as desired.

We recall that an addition of solutions is again a solution. So that means, as we have established already that each $u_n(x,t)$ is a nontrivial solution for $n = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} u_n(x,y) \\ &= \sum_{n=1}^{\infty} \sin(nx)(A_ne^{ny} + B_ne^{-ny}) \end{aligned}$$

is also a nontrivial solution of the problem. Next, we will compute the Fourier coefficients A_n, B_n. We have

$$\begin{aligned} 1 = u(x,0) &= \sum_{n=1}^{\infty} \sin(nx)(A_n + B_n), \\ 1 = u(x,\pi) &= \sum_{n=1}^{\infty} \sin(nx)(A_ne^{n\pi} + B_ne^{-n\pi}). \end{aligned}$$

Now, recall

$$\int_0^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} \frac{\pi}{2} & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Consequently, we obtain

$$\begin{aligned} \int_0^{\pi} 1\sin(nx)\,dx &= \int_0^{\pi} \sum_{m=1}^{\infty} \sin(mx)(A_m + B_m)\sin(nx)\,dx \\ &= \sum_{m=1}^{\infty} (A_m + B_m)\int_0^{\pi} \sin(mx)\sin(nx)\,dx \\ &= (A_n + B_n)\frac{\pi}{2} \end{aligned}$$

and

$$\begin{aligned} \int_0^{\pi} 1\sin(nx)\,dx &= \int_0^{\pi} \sum_{m=1}^{\infty} \sin(mx)(A_me^{m\pi} + B_me^{-m\pi})\sin(nx)\,dx \\ &= \sum_{m=1}^{\infty} (A_me^{m\pi} + B_me^{-m\pi})\int_0^{\pi} \sin(mx)\sin(nx)\,dx \\ &= (A_ne^{n\pi} + B_ne^{-n\pi})\frac{\pi}{2}. \end{aligned}$$

In other words, we have the system

$$A_n + B_n = \frac{2}{\pi}\int_0^\pi 1\sin(nx)\,dx = \begin{cases} \frac{4}{\pi}\frac{1}{n} & \text{if } n = 1,3,5,\dots \\ 0 & \text{if } n = 2,4,6,\dots, \end{cases}$$

$$A_n e^{n\pi} + B_n e^{-n\pi} = \frac{2}{\pi}\int_0^\pi 1\sin(nx)\,dx = \begin{cases} \frac{4}{\pi}\frac{1}{n} & \text{if } n = 1,3,5,\dots \\ 0 & \text{if } n = 2,4,6,\dots, \end{cases}$$

which we can solve simultaneously to obtain the coefficients

$$A_n = \begin{cases} \frac{2}{\pi}\frac{1}{n\sinh(n\pi)}(1 - e^{-n\pi}) & \text{if } n = 1,3,5,\dots, \\ 0 & \text{if } n = 2,4,6,\dots, \end{cases}$$

$$B_n = \begin{cases} \frac{2}{\pi}\frac{1}{n\sinh(n\pi)}(e^{n\pi} - 1) & \text{if } n = 1,3,5,\dots, \\ 0 & \text{if } n = 2,4,6,\dots. \end{cases}$$

So our formal solution is

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} \sin(nx)(A_n e^{ny} + B_n e^{-ny}) \\ &= \sum_{n=1,3,5,\dots} \sin(nx)\left(\frac{2}{\pi}\frac{1}{n\sinh(n\pi)}(1 - e^{-n\pi})e^{ny} + \frac{2}{\pi}\frac{1}{n\sinh(n\pi)}(e^{n\pi} - 1)e^{-ny}\right) \\ &= \frac{2}{\pi}\sum_{n=1,3,5,\dots} \frac{1}{n\sinh(n\pi)}\sin(nx)((1 - e^{-n\pi})e^{ny} + (e^{n\pi} - 1)e^{-ny}) \\ &= \frac{4}{\pi}\sum_{n=1,3,5,\dots} \frac{1}{n\sinh(n\pi)}\sin(nx)\left(\frac{e^{n(\pi-y)} - e^{-n(\pi-y)}}{2} + \frac{e^{ny} - e^{-ny}}{2}\right) \\ &= \frac{4}{\pi}\sum_{n=1,3,5,\dots} \frac{1}{n\sinh(n\pi)}\sin(nx)(\sinh(n(\pi - y)) + \sinh(ny)) \\ &= \boxed{\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)\sinh((2n-1)\pi)}\sin((2n-1)x)(\sinh((2n-1)(\pi - y)) + \sinh((2n-1)y))}. \end{aligned}$$

as desired. □

7.7. (a) Compute the Laplace equation in a polar coordinate system.

Solution. We know already that the Laplacian is defined in the Cartesian coordinate system by

$$\Delta u = u_{xx} + u_{yy}.$$

To compute the Laplace equation $\Delta u = 0$ in the polar coordinate system, we need to derive the equivalent expression of the Laplacian in polar coordinates. Let

$$\begin{aligned} x &= x(r,\theta) = r\cos(\theta), \\ y &= y(r,\theta) = r\sin(\theta), \\ u(x,y) &= w(r,\theta) = u(x(r,\theta), y(r,\theta)), \end{aligned}$$

the first two of which imply

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right). \end{aligned}$$

We obtain first partial derivatives

$$\begin{aligned} r_x &= (\sqrt{x^2 + y^2})_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}, \\ r_y &= (\sqrt{x^2 + y^2})_y = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r}, \\ \theta_x &= \left(\tan^{-1}\left(\frac{y}{x}\right)\right)_x = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2}, \\ \theta_y &= \left(\tan^{-1}\left(\frac{y}{x}\right)\right)_y = \frac{x}{x^2 + y^2} = \frac{x}{r^2} \end{aligned}$$

and the second partial derivatives

$$
\begin{aligned}
r_{xx} &= \left(\frac{x}{\sqrt{x^2+y^2}}\right)_x = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{y^2}{r^3},\\
r_{yy} &= \left(\frac{y}{\sqrt{x^2+y^2}}\right)_y = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{x^2}{r^3},\\
\theta_{xx} &= \left(-\frac{y}{x^2+y^2}\right)_x = \frac{2xy}{(x^2+y^2)^2} = \frac{2xy}{r^4},\\
\theta_{yy} &= \left(\frac{x}{x^2+y^2}\right)_y = -\frac{2xy}{(x^2+y^2)^2} = -\frac{2xy}{r^4}.
\end{aligned}
$$

So, by the multivariable chain rule, we obtain the second partial derivatives

$$
\begin{aligned}
u_{xx} &= (w(r,\theta))_{xx}\\
&= (w_r r_x + w_\theta \theta_x)_x\\
&= (w_r r_x)_x + (w_\theta \theta_x)_x\\
&= (w_{rr}(r_x)^2 + w_r r_{xx}) + (w_{\theta\theta}(\theta_x)^2 + w_\theta \theta_{xx})\\
&= w_{rr}\frac{x^2}{r^2} + w_r\frac{y^2}{r^3} + w_{\theta\theta}\frac{y^2}{r^4} + w_\theta\frac{2xy}{r^4}
\end{aligned}
$$

and

$$
\begin{aligned}
u_{yy} &= (w(r,\theta))_{yy}\\
&= (w_r r_x + w_\theta \theta_x)_y\\
&= (w_r r_y)_y + (w_\theta \theta_y)_y\\
&= (w_{rr}(r_y)^2 + w_r r_{yy}) + (w_{\theta\theta}(\theta_y)^2 + w_\theta \theta_{yy})\\
&= w_{rr}\frac{y^2}{r^2} + w_r\frac{x^2}{r^3} + w_{\theta\theta}\frac{x^2}{r^4} - w_\theta\frac{2xy}{r^4}.
\end{aligned}
$$

Therefore, the Laplacian in polar coordinates is

$$
\begin{aligned}
\Delta u &= u_{xx} + u_{yy}\\
&= \left(w_{rr}\frac{x^2}{r^2} + w_r\frac{y^2}{r^3} + w_{\theta\theta}\frac{y^2}{r^4} + \cancel{w_\theta\frac{2xy}{r^4}}\right) + \left(w_{rr}\frac{y^2}{r^2} + w_r\frac{x^2}{r^3} + w_{\theta\theta}\frac{x^2}{r^4} - \cancel{w_\theta\frac{2xy}{r^4}}\right)\\
&= w_{rr}\frac{x^2+y^2}{r^2} + w_r\frac{x^2+y^2}{r^3} + w_{\theta\theta}\frac{x^2+y^2}{r^4}\\
&= w_{rr}\frac{r^2}{r^2} + w_r\frac{r^2}{r^3} + w_{\theta\theta}\frac{r^2}{r^4}\\
&= w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta}.
\end{aligned}
$$

This means that the Laplace equation $\Delta u = 0$ in polar coordinates is written

$$
w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} = 0,
$$

as desired. □

(b) Let $D \subset \mathbb{R}^2$ be the disk $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 6\}$. Solve the problem

$$
\begin{aligned}
\Delta u &= 0 && (x,y) \in D,\\
u(x,y) &= y + y^2 && (x,y) \in \partial D.
\end{aligned}
$$

Write your answer in the Cartesian coordinate system.

Solution. Define $w(r,\theta) = u(x(r,\theta), y(r,\theta))$. Then the problem is transformed into

$$
\begin{aligned}
\Delta w &= 0 && 0 < r < \sqrt{6}, 0 \le \theta \le 2\pi\\
w(\sqrt{6},\theta) &= \sqrt{6}\sin(\theta) + 6\sin^2(\theta) && 0 \le \theta \le 2\pi
\end{aligned}
$$

We want to find a solution of the form

$$
w(r,\theta) = R(r)\Theta(\theta).
$$

Our partial derivatives are

$$\begin{aligned} w_r(r,\theta) &= R_r(r)\Theta(\theta), \\ w_{rr}(r,\theta) &= R_{rr}(r)\Theta(\theta), \\ w_{\theta\theta}(r,\theta) &= R(r)\Theta_{\theta\theta}(\theta). \end{aligned}$$

So the partial differential equation

$$w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} = \Delta w = 0$$

becomes

$$R_{rr}(r)\Theta(\theta) + \frac{1}{r}R_r(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta_{\theta\theta}(\theta) = 0,$$

which we can algebraically rearrange to write

$$-\frac{r^2R_{rr}(r) + rR_r(r)}{R(r)} = \frac{\Theta_{\theta\theta}(\theta)}{\Theta(\theta)} = -\lambda,$$

where λ is a constant in both r and θ. This produces the system of two ordinary differential equations

$$\begin{aligned} r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda R &= 0 \\ \frac{d^2\Theta}{dr^2} + \lambda\Theta &= 0. \end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$R(r) = \begin{cases} C_1\cos(\sqrt{-\lambda}\ln(r)) + C_2\sin(\sqrt{-\lambda}\ln(r)) & \text{if } \lambda < 0, \\ C_1\ln(r) + C_2 & \text{if } \lambda = 0, \\ C_1 r^{\sqrt{\lambda}} + C_2 r^{-\sqrt{\lambda}} & \text{if } \lambda > 0, \end{cases}$$

$$\Theta(\theta) = \begin{cases} D_1 e^{\sqrt{-\lambda}\theta} + D_2 e^{-\sqrt{-\lambda}\theta} & \text{if } \lambda < 0, \\ D_1\theta + D_2 & \text{if } \lambda = 0, \\ D_1\cos(\sqrt{\lambda}\theta) + D_2\sin(\sqrt{\lambda}\theta) & \text{if } \lambda > 0, \end{cases}$$

where C_1, C_2, D_1, D_2 are constants. Now, according to page 196 of the textbook, the equation for Θ holds at the interval $(0, 2\pi)$. In order for $\Theta(\theta)$ to be twice differentiable (so that $\frac{d^2\Theta}{d\theta^2}$ makes sense, after all) for all $\theta \in \mathbb{R}$, we need to impose the periodic boundary conditions

$$\begin{aligned} \Theta(0) &= \Theta(2\pi), \\ \frac{d}{d\theta}\Theta(0) &= \frac{d}{d\theta}\Theta(2\pi), \end{aligned}$$

which will impose constraints on the constants D_1, D_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then we have

  $$\begin{aligned} \Theta(\theta) &= D_1 e^{\sqrt{-\lambda}\theta} + D_2 e^{-\sqrt{-\lambda}\theta}, \\ \Theta(0) &= \Theta(2\pi), \end{aligned}$$

  which implies $D_1 + D_2 = D_1 e^{2\pi\sqrt{-\lambda}} + D_2 e^{-2\pi\sqrt{-\lambda}}$. We also have

  $$\begin{aligned} \frac{d}{d\theta}\Theta(\theta) &= \sqrt{-\lambda}(D_1 e^{\sqrt{-\lambda}x} - D_2 e^{-\sqrt{-\lambda}x}), \\ \frac{d}{d\theta}\Theta(0) &= \frac{d}{d\theta}\Theta(2\pi), \end{aligned}$$

  which implies $D_1 - D_2 = D_1 e^{2\pi\sqrt{-\lambda}} - D_2 e^{-2\pi\sqrt{-\lambda}}$. Now we will solve for the constants D_1, D_2. We have formulated the linear system

  $$\begin{aligned} D_1 + D_2 &= D_1 e^{2\pi\sqrt{-\lambda}} + D_2 e^{-2\pi\sqrt{-\lambda}}, \\ D_1 - D_2 &= D_1 e^{2\pi\sqrt{-\lambda}} - D_2 e^{-2\pi\sqrt{-\lambda}}, \end{aligned}$$

  and we can algebraically rearrange each equation in the system to write

  $$\begin{aligned} D_1(1 - e^{2\pi\sqrt{-\lambda}}) &= -D_2(1 - e^{-2\pi\sqrt{-\lambda}}), \\ D_1(1 - e^{2\pi\sqrt{-\lambda}}) &= D_2(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

We can combine the two equations in the system to deduce

$$\begin{aligned} D_1(1 - e^{2\pi\sqrt{-\lambda}}) &= -D_2(1 - e^{-2\pi\sqrt{-\lambda}}) \\ &= -D_1(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

Since we are currently in the case $\lambda < 0$, we have $1 - e^{2\pi\sqrt{-\lambda}} \neq 0$, and so we can divide $1 - e^{2\pi\sqrt{-\lambda}}$ from both sides of our previous equation to conclude $C_1 = -C_1$, or $C_1 = 0$. Likewise, we can combine the two equations in the system to deduce

$$\begin{aligned} D_2(1 - e^{-2\pi\sqrt{-\lambda}}) &= D_1(1 - e^{2\pi\sqrt{-\lambda}}) \\ &= -D_2(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

Since we are currently in the case $\lambda < 0$, we have $1 - e^{-2\pi\sqrt{-\lambda}} \neq 0$, and so we can divide $1 - e^{-2\pi\sqrt{-\lambda}}$ from both sides of our previous equation to conclude $D_2 = -D_2$, or $D_2 = 0$. So we have

$$\begin{aligned} \Theta(\theta) &= C_1 e^{\sqrt{-\lambda}\theta} + C_2 e^{-\sqrt{-\lambda}\theta} \\ &= 0e^{\sqrt{-\lambda}\theta} + 0e^{-\sqrt{-\lambda}\theta} \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} w(r, \theta) &= R(r)\Theta(\theta) \\ &= R(r) \cdot 0 \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 2: Suppose $\lambda = 0$. Then we have

$$\begin{aligned} &\Theta(\theta) = D_1\theta + D_2, \\ &\Theta(0) = \Theta(2\pi), \end{aligned}$$

which implies $D_1 = 0$, and so we have

$$\begin{aligned} \Theta(\theta) &= D_1\theta + D_2 \\ &= D_1 \cdot 0 + D_2 \\ &= D_2. \end{aligned}$$

The derivative is

$$\begin{aligned} \frac{d}{d\theta}\Theta(\theta) &= \frac{d}{d\theta}(D_2) \\ &= 0, \end{aligned}$$

which clearly satisfies $\frac{d}{d\theta}\Theta(0) = 0 = \frac{d}{d\theta}\Theta(2\pi)$. Next, observe that $\ln(r)$ is undefined at the origin (at $r = 0$). Following page 197 of the textbook, we only consider smooth solutions and disregard any solutions that are undefined at the origin, and so we shall impose the condition $C_1 = 0$. So we have

$$\begin{aligned} R(r) &= C_1 \ln(r) + C_2 \\ &= 0 \ln(r) + C_2 \\ &= C_2. \end{aligned}$$

Therefore, if we write $\frac{\alpha_0}{2} = C_2 D_2$, then we have

$$\begin{aligned} w_0(r, \theta) &= R(r)\Theta(\theta) \\ &= C_2 D_2 \\ &= \frac{\alpha_0}{2}, \end{aligned}$$

which is a nontrivial smooth solution on a disk.

- Case 3: Suppose $\lambda > 0$. Then we have

$$\begin{aligned} &\Theta(\theta) = D_1 \cos(\sqrt{\lambda}\theta) + D_2 \sin(\sqrt{\lambda}\theta), \\ &\Theta(0) = \Theta(2\pi), \end{aligned}$$

which implies

$$D_1 = D_1 \cos(2\pi\sqrt{\lambda}) + D_2 \sin(2\pi\sqrt{\lambda}). \tag{1}$$

We also have

$$\begin{aligned} \frac{d}{d\theta}\Theta(\theta) &= \sqrt{\lambda}(-D_1 \sin(\sqrt{\lambda}\theta) + D_2 \cos(\sqrt{\lambda}\theta)), \\ \frac{d}{d\theta}\Theta(0) &= \frac{d}{d\theta}\Theta(2\pi), \end{aligned}$$

which implies

$$D_2 = -D_1 \sin(2\pi\sqrt{\lambda}) + D_2 \cos(2\pi\sqrt{\lambda}). \tag{2}$$

Now, we claim that, if either $\sin(2\pi\sqrt{\lambda}) \neq 0$ or $\cos(2\pi\sqrt{\lambda}) \neq 1$, then we have $D_1 = 0$ and $D_2 = 0$.

- Subcase 1: Suppose $\sin(2\pi\sqrt{\lambda}) \neq 0$. Multiply both sides of (1) by $-\cos(2\pi\sqrt{\lambda})$ and both sides of (2) by $\sin(2\pi\sqrt{\lambda})$ to obtain

  $$\begin{aligned} -D_1 \cos(2\pi\sqrt{\lambda}) &= -D_1 \cos^2(2\pi\sqrt{\lambda}) - D_2 \sin(2\pi\sqrt{\lambda}) \cos(2\pi\sqrt{\lambda}), \\ D_2 \sin(2\pi\sqrt{\lambda}) &= -D_1 \sin^2(2\pi\sqrt{\lambda}) + D_2 \cos(2\pi\sqrt{\lambda}) \sin(2\pi\sqrt{\lambda}), \end{aligned}$$

  from which we can add up both sides of the two equations to get

  $$-D_1 \cos(2\pi\sqrt{\lambda}) + D_2 \sin(2\pi\sqrt{\lambda}) = -D_1. \tag{3}$$

  We equate (1) and (3) to get

  $$\cancel{D_1 \cos(2\pi\sqrt{\lambda})} - D_2 \sin(2\pi\sqrt{\lambda}) = \cancel{D_1 \cos(2\pi\sqrt{\lambda})} + D_2 \sin(2\pi\sqrt{\lambda}),$$

  which simplifies to

  $$-D_2 \cancel{\sin(2\pi\sqrt{\lambda})} = D_2 \cancel{\sin(2\pi\sqrt{\lambda})}.$$

  Since we assumed $\sin(2\pi\sqrt{\lambda}) \neq 0$, we can divide both sides by $\sin(2\pi\sqrt{\lambda})$ to get $-D_2 = D_2$, which means $D_2 = 0$. Substitute $D_2 = 0$ into (2) to obtain

  $$0 = -D_1 \sin(2\pi\sqrt{\lambda}),$$

  which implies $D_1 = 0$ because, once again, we assumed $\sin(2\pi\sqrt{\lambda}) \neq 0$.
- Subcase 2: Suppose $\cos(2\pi\sqrt{\lambda}) \neq 1$. Then we can rewrite (1) and (2) as

  $$D_1(1 - \cos(2\pi\sqrt{\lambda})) = D_2 \sin(2\pi\sqrt{\lambda}), \tag{4}$$

  $$D_2(1 - \cos(2\pi\sqrt{\lambda})) = -D_1 \sin(2\pi\sqrt{\lambda}),. \tag{5}$$

  Multiply both sides of (4) by D_1 and both sides of (5) by D_2 to obtain

  $$\begin{aligned} D_1^2(1 - \cos(2\pi\sqrt{\lambda})) &= D_1 D_2 \sin(2\pi\sqrt{\lambda}), \\ D_2^2(1 - \cos(2\pi\sqrt{\lambda})) &= -D_1 D_2 \sin(2\pi\sqrt{\lambda}), \end{aligned}$$

  from which we can add up both sides of the two equations to get

  $$(D_1^2 + D_2^2)(1 - \cos(2\pi\sqrt{\lambda})) = 0.$$

  Since we assumed $\cos(2\pi\sqrt{\lambda}) \neq 1$, we must conclude $D_1^2 + D_2^2 = 0$, which forces $D_1 = 0$ and $D_2 = 0$.

So we have proved our claim. Now that we have established our claim, we would have

$$\begin{aligned} \Theta(\theta) &= D_1 \cos(\sqrt{\lambda}\theta) + D_2 \sin(\sqrt{\lambda}\theta) \\ &= 0 \cos(\sqrt{\lambda}\theta) + 0 \sin(\sqrt{\lambda}\theta) \\ &= 0, \end{aligned}$$

which would imply that $w(r, \theta) = R(r)\Theta(\theta)$ is a trivial solution. Therefore, to find a nontrivial solution for this case, we should assume both

$$\begin{aligned} \sin(2\pi\sqrt{\lambda}) &= 0, \\ 1 - \cos(2\pi\sqrt{\lambda}) &= 0, \end{aligned}$$

which imply $2\pi\sqrt{\lambda} = 2n\pi$, or equivalently

$$\lambda_n = \lambda = n^2,$$

and so we have

$$\begin{aligned}\Theta_n(\theta) &= D_{1,n}\cos(\sqrt{\lambda_n}\theta) + D_{2,n}\sin(\sqrt{\lambda_n}\theta)\\ &= D_{1,n}\cos(\sqrt{n^2}\theta) + D_{2,n}\sin(\sqrt{n^2}\theta)\\ &= D_{1,n}\cos(n\theta) + D_{2,n}\sin(n\theta)\end{aligned}$$

and

$$\begin{aligned}R_n(r) &= C_{1,n}r^{\sqrt{\lambda_n}} + C_{2,n}r^{-\sqrt{\lambda_n}}\\ &= C_{1,n}r^{\sqrt{n^2}} + C_{2,n}r^{-\sqrt{n^2}}\\ &= C_{1,n}r^n + C_{2,n}r^{-n}\end{aligned}$$

for $n = 1, 2, 3, \ldots$. Observe that r^{-n} for $n = 1, 2, \ldots$ is undefined at the origin (at $r = 0$). Following page 197 of the textbook, we only consider smooth solutions and disregard any solutions that are undefined at the origin, and so we shall impose the condition $C_{2,n} = 0$. So we have

$$\begin{aligned}R_n(r) &= C_{1,n}r^n + C_{2,n}r^{-n}\\ &= C_{1,n}r^n + 0r^{-n}\\ &= C_{1,n}r^n.\end{aligned}$$

Therefore, if we write $\alpha_n := C_{1,n}D_{1,n}$ and $\beta_n := C_{1,n}D_{2,n}$, then we have

$$\begin{aligned}w_n(r,\theta) &= R_n(r)\Theta_n(\theta)\\ &= (C_{1,n}r^n)(D_{1,n}\cos(n\theta) + D_{2,n}\sin(n\theta))\\ &= r^n(C_{1,n}D_{1,n}\cos(n\theta) + C_{1,n}D_{2,n}\sin(n\theta))\\ &= r^n(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta)).\end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial smooth solution on a disk.

We recall that an addition of smooth solutions is again a smooth solution. So that means, as we have established already that each $w_n(r,\theta)$ is a nontrivial smooth solution for $n = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned}w(r,\theta) &= w_0(r,\theta) + \sum_{n=1}^{\infty} w_n(r,\theta)\\ &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta))\end{aligned}$$

is a general smooth solution of the Laplace equation on a disk. Next, we will compute the Fourier coefficients $\alpha_0, \alpha_n, \beta_n$. We have

$$\begin{aligned}w(\sqrt{6},\theta) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} 6^{\frac{n}{2}}(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta))\\ &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} 6^{\frac{n}{2}}\alpha_n\cos(n\theta) + \sum_{n=1}^{\infty} 6^{\frac{n}{2}}\beta_n\sin(n\theta)\end{aligned}$$

and the given boundary condition

$$\begin{aligned}w(\sqrt{6},\theta) &= \sqrt{6}\sin(\theta) + 6\sin^2(\theta)\\ &= \sqrt{6}\sin(\theta) + 3(1 - \cos(2\theta))\\ &= 3 - 3\cos(2\theta) + \sqrt{6}\sin(\theta).\end{aligned}$$

Both our expressions of $w(\sqrt{6},\theta)$ yield

$$\frac{\alpha_0}{2} + \sum_{n=1}^{\infty} 6^{\frac{n}{2}}\alpha_n\cos(n\theta) + \sum_{n=1}^{\infty} 6^{\frac{n}{2}}\beta_n\sin(n\theta) = 3 - 3\cos(2\theta) + \sqrt{6}\sin(\theta).$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to find the Fourier coefficients

$$\begin{aligned}\alpha_0 &= 6,\\ \alpha_2 &= -\frac{1}{2},\\ \alpha_n &= 0\end{aligned}$$

for $n = 1$ and for $n = 3, 4, 5, \ldots$ and

$$\beta_1 = 1, \quad \beta_n = 0$$

for $n = 2, 3, 4, \ldots$. Therefore, our formal solution in polar coordinates is

$$\begin{aligned}
w(r,\theta) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n(\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)) \\
&= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} \beta_n r^n \sin(n\theta) \\
&= \frac{\alpha_0}{2} + \left(\alpha_2 r^2 \cos(2\theta) + \sum_{\substack{n=1\\ n=3,4,5,\ldots}} \alpha_n r^n \cos(n\theta)\right) + \left(\beta_1 r^1 \sin(1\theta) + \sum_{n=2,3,4,\ldots} \beta_n r^n \sin(n\theta)\right) \\
&= \frac{6}{2} + \left(-\frac{1}{2} r^2 \cos(2\theta) + \sum_{\substack{n=1\\ n=3,4,5,\ldots}} 0 r^n \cos(n\theta)\right) + \left(1 r^1 \sin(1\theta) + \sum_{n=2,3,4,\ldots} 0 r^n \sin(n\theta)\right) \\
&= \boxed{3 - \frac{1}{2} r^2 \cos(2\theta) + r\sin(\theta)}.
\end{aligned}$$

In Cartesian coordinates, our formal solution is

$$\begin{aligned}
u(x,y) &= u(x(r,\theta), y(r,\theta)) \\
&= w(r,\theta) \\
&= 3 - \frac{1}{2} r^2 \cos(2\theta) + r\sin(\theta) \\
&= 3 - \frac{1}{2} r^2 (\cos^2(\theta) - \sin^2(\theta)) + r\sin(\theta) \\
&= 3 - \frac{1}{2} ((r\cos(\theta))^2 - (r\sin(\theta))^2) + r\sin(\theta) \\
&= \boxed{3 - \frac{1}{2}(x^2 - y^2) + y},
\end{aligned}$$

where we used $x = r\cos(\theta)$ and $y = r\sin(\theta)$. □

7.8. (a) Solve the problem

$$\begin{aligned}
\Delta u &= 0 && 0 < x < \pi, 0 < y < \pi, \\
u(x,0) = u(x,\pi) &= 0 && 0 \le x \le \pi, \\
u(0,y) = 0, u(\pi,y) &= \sin(y) && 0 \le y \le \pi.
\end{aligned}$$

Solution. We want to find a solution of the form

$$u(x,y) = X(x)Y(y).$$

Our partial derivatives are

$$\begin{aligned}
u_{xx}(x,t) &= X_{xx}(x)Y(y), \\
u_{yy}(x,t) &= X(x)Y_{yy}(y)
\end{aligned}$$

So the partial differential equation

$$u_{xx} + u_{yy} = \Delta u = 0$$

becomes

$$X_{xx}(x)Y(y) + X(x)Y_{yy}(y) = 0,$$

which we can algebraically rearrange to write

$$-\frac{X_{xx}(x)}{X(x)} = \frac{Y_{yy}(y)}{Y(y)} = -\lambda,$$

where λ is a constant in both x and y. This produces the system of two ordinary differential equations

$$\begin{aligned}
\frac{d^2X}{dx^2} - \lambda X &= 0 \\
\frac{d^2Y}{dy^2} + \lambda Y &= 0.
\end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$X(x) = \begin{cases} C_1 \cos(\sqrt{-\lambda}x) + C_2 \sin(\sqrt{-\lambda}x) & \text{if } \lambda < 0, \\ C_1 x + C_2 & \text{if } \lambda = 0, \\ C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x} & \text{if } \lambda > 0, \end{cases}$$

$$Y(y) = \begin{cases} D_1 e^{\sqrt{-\lambda}y} + D_2 e^{-\sqrt{-\lambda}y} & \text{if } \lambda < 0, \\ D_1 y + D_2 & \text{if } \lambda = 0, \\ D_1 \cos(\sqrt{\lambda}y) + D_2 \sin(\sqrt{\lambda}y) & \text{if } \lambda > 0, \end{cases}$$

where C_1, C_2, D_1, D_2 are constants. Now, the boundary conditions

$$u(0, y) = u(x, 0) = u(x, \pi) = 0$$

are equivalent to

$$X(0)Y(y) = X(x)Y(0) = X(x)Y(\pi) = 0,$$

which imply either $X(x) = Y(y) = 0$ or $X(\pi) = Y(0) = Y(\pi) = 0$. If we assume either $X(x) = 0$ or $Y(y) = 0$, then in either case we would have a trivial solution. But we are really interested in finding a nontrivial solution. So we should assume

$$X(0) = Y(0) = Y(\pi) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then

  $$\begin{aligned} Y(y) &= D_1 e^{\sqrt{-\lambda}y} + D_2 e^{-\sqrt{-\lambda}y}, \\ Y(0) &= 0 \end{aligned}$$

  implies $D_2 = -D_1$, and so we have

  $$\begin{aligned} Y(y) &= D_1 e^{\sqrt{-\lambda}y} + D_2 e^{-\sqrt{-\lambda}y} \\ &= D_1 e^{\sqrt{-\lambda}y} - D_1 e^{-\sqrt{-\lambda}y} \\ &= D_1 (e^{\sqrt{-\lambda}y} - e^{-\sqrt{-\lambda}y}). \end{aligned}$$

  Now, if $\lambda < 0$, then $e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi} \neq 0$. This means

  $$\begin{aligned} Y(y) &= D_1 (e^{\sqrt{-\lambda}y} - D_2 e^{-\sqrt{-\lambda}y}), \\ Y(\pi) &= 0 \end{aligned}$$

  implies $D_1 = 0$, and so we have

  $$\begin{aligned} X(x) &= D_1 (e^{\sqrt{-\lambda}y} - e^{-\sqrt{-\lambda}y}) \\ &= 0 (e^{\sqrt{-\lambda}y} - e^{-\sqrt{-\lambda}y}) \\ &= 0. \end{aligned}$$

  Therefore, we have

  $$\begin{aligned} u(x, y) &= X(x)Y(y) \\ &= 0Y(y) \\ &= 0, \end{aligned}$$

  which is a trivial solution.
- Case 2: Suppose $\lambda = 0$. Then

  $$\begin{aligned} Y(y) &= D_1 y + D_2, \\ Y(0) &= 0 \end{aligned}$$

  implies $D_2 = 0$, and so we have

  $$\begin{aligned} Y(y) &= D_1 y + D_2 \\ &= D_1 y + 0 \\ &= D_1 y. \end{aligned}$$

Next,

$$\begin{aligned} Y(y) &= D_1 y, \\ Y(\pi) &= 0 \end{aligned}$$

implies $D_1 = 0$, and so we have

$$\begin{aligned} Y(y) &= D_1 y \\ &= 0y \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u(x, y) &= X(x)Y(y) \\ &= 0Y(y) \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 3: Suppose $\lambda > 0$. Then

$$\begin{aligned} Y(y) &= D_1 \cos(\sqrt{\lambda} y) + D_2 \sin(\sqrt{\lambda} y), \\ Y(0) &= 0 \end{aligned}$$

implies $D_1 = 0$, and so we have

$$\begin{aligned} Y(y) &= D_1 \cos(\sqrt{\lambda} y) + D_2 \sin(\sqrt{\lambda} y) \\ &= 0 \cos(\sqrt{\lambda} x) + D_2 \sin(\sqrt{\lambda} y) \\ &= D_2 \sin(\sqrt{\lambda} y). \end{aligned}$$

Next,

$$\begin{aligned} Y(y) &= D_2 \sin(\sqrt{\lambda} y), \\ Y(\pi) &= 0 \end{aligned}$$

implies $\sin(\sqrt{\lambda} y) = 0$, which in turn implies $\sqrt{\lambda}\pi = n\pi$, or equivalently

$$\lambda_n = \lambda = n^2.$$

Also,

$$\begin{aligned} X(x) &= C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}, \\ X(0) &= 0 \end{aligned}$$

implies $C_2 = -C_1$, and so we have

$$\begin{aligned} X(x) &= C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x} \\ &= C_1 e^{\sqrt{\lambda} x} - C_1 e^{-\sqrt{\lambda} x} \\ &= 2C_1 \frac{e^{\sqrt{\lambda} x} - e^{-\sqrt{\lambda} x}}{2} \\ &= 2C_1 \sinh(\sqrt{\lambda} x). \end{aligned}$$

Therefore, we have

$$\begin{aligned} X_n(x) &= 2C_{1,n} \sinh(\sqrt{\lambda_n} x) \\ &= 2C_{1,n} \sinh(\sqrt{n^2} x) \\ &= 2C_{1,n} \sinh(nx). \end{aligned}$$

and

$$\begin{aligned} Y_n(y) &= D_{2,n} \sin(\sqrt{\lambda_n} y) \\ &= D_{2,n} \sin(\sqrt{n^2} y) \\ &= D_{2,n} \sin(ny) \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Therefore, if we write $A_n = 2C_{1,n}D_{1,n}$, then we have

$$\begin{aligned} u_n(x, y) &= X_n(x)Y_n(y) \\ &= (2C_{1,n} \sinh(nx))(D_{2,n} \sin(ny)) \\ &= 2C_{1,n}D_{2,n} \sinh(nx) \sin(ny) \\ &= A_n \sinh(nx) \sin(ny) \end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial solution, as desired.

We recall that an addition of solutions is again a solution. So that means, as we have established already that each $u_n(x, t)$ is a nontrivial solution for $n = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} u_n(x, y) \\ &= \sum_{n=1}^{\infty} A_n \sinh(nx) \sin(ny) \end{aligned}$$

is also a nontrivial solution of the problem. Next, we will compute the Fourier coefficients A_n. We have

$$\sin(y) = u(\pi, y) = \sum_{n=1}^{\infty} A_n \sinh(n\pi) \sin(ny).$$

Now, recall

$$\int_0^{\pi} \sin(ny) \sin(my)\, dy = \begin{cases} \frac{\pi}{2} & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Consequently, we obtain

$$\begin{aligned} \int_0^{\pi} \sin(y) \sin(ny)\, dy &= \int_0^{\pi} \sum_{m=1}^{\infty} A_m \sinh(m\pi) \sin(my) \sin(ny)\, dy \\ &= \sum_{m=1}^{\infty} A_m \sinh(m\pi) \int_0^{\pi} \sin(my) \sin(ny)\, dy \\ &= A_n \sinh(n\pi) \frac{\pi}{2}, \end{aligned}$$

which implies

$$\begin{aligned} A_n &= \frac{2}{\pi \sinh(n\pi)} \int_0^{\pi} \sin(y) \sin(ny)\, dy \\ &= \begin{cases} \frac{1}{\sinh(n\pi)} & \text{if } n = 1, \\ 0 & \text{if } n = 2, 3, 4, \ldots. \end{cases} \end{aligned}$$

So our formal solution is

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} A_n \sinh(nx) \sin(ny) \\ &= A_1 \sinh(1x) \sin(1y) + \sum_{n=2}^{\infty} A_n \sinh(nx) \sin(ny) \\ &= \frac{1}{\sinh(\pi)} \sinh(1x) \sin(1y) + \sum_{n=2}^{\infty} 0 \sinh(nx) \sin(ny) \\ &= \boxed{\frac{1}{\sinh(\pi)} \sinh(x) \sin(y)}, \end{aligned}$$

as desired. □

(b) Is there a point $\{(x, y) \in \mathbb{R}^2 \mid 0 < x < \pi, 0 < y < \pi\}$ such that $u(x, y) = 0$?

Answer. No, there does not exist a point in $\{(x, y) \in \mathbb{R}^2 \mid 0 < x < \pi, 0 < y < \pi\}$ such that $u(x, y) = 0$. Because we have $\frac{d}{dx}(\sinh(x)) = \cosh(x) > 0$ for all $0 < x < \pi$, it follows that $\sinh(x)$ is an increasing function of x, which implies in particular $\sinh(x) > \sinh(0) = 0$ for all $0 < x < \pi$. Also, we have $\sin(y) > 0$ for all $0 < y < \pi$. So we conclude

$$u(x, y) = \frac{1}{\sinh(\pi)} \sinh(x) \sin(y) > 0,$$

which implies $u(x, y) \neq 0$ on all of $\{(x, y) \in \mathbb{R}^2 \mid 0 < x < \pi, 0 < y < \pi\}$. □

7.11. Let $D \subset \mathbb{R}^2$ be the domain $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 4\}$. Solve

$$\begin{aligned} \Delta u &= 0 \quad (x, y) \in D, \\ u(x, y) &= y \quad (x, y) \in \partial D, \\ \lim_{|x|+|y|\to\infty} u(x, y) &= 0. \end{aligned}$$

Solution. Define $w(r, \theta) = u(x(r, \theta), y(r, \theta))$. Then the problem is transformed into

$$\begin{aligned} \Delta w &= 0 && 0 < r < \sqrt{6}, 0 \le \theta \le 2\pi \\ w(2, \theta) &= 2\sin(\theta) && 0 \le \theta \le 2\pi \\ \lim_{r\to\infty} w(r, \theta) &= 0. \end{aligned}$$

We want to find a solution of the form

$$w(r, \theta) = R(r)\Theta(\theta).$$

Our partial derivatives are

$$\begin{aligned} w_r(r, \theta) &= R_r(r)\Theta(\theta), \\ w_{rr}(r, \theta) &= R_{rr}(r)\Theta(\theta), \\ w_{\theta\theta}(r, \theta) &= R(r)\Theta_{\theta\theta}(\theta). \end{aligned}$$

So the partial differential equation

$$w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} = \Delta w = 0$$

becomes

$$R_{rr}(r)\Theta(\theta) + \frac{1}{r}R_r(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta_{\theta\theta}(\theta) = 0,$$

which we can algebraically rearrange to write

$$-\frac{r^2R_{rr}(r) + rR_r(r)}{R(r)} = \frac{\Theta_{\theta\theta}(\theta)}{\Theta(\theta)} = -\lambda,$$

where λ is a constant in both r and θ. This produces the system of two ordinary differential equations

$$\begin{aligned} r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda R &= 0 \\ \frac{d^2\Theta}{dr^2} + \lambda\Theta &= 0. \end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$R(r) = \begin{cases} C_1 \cos(\sqrt{-\lambda}\ln(r)) + C_2 \sin(\sqrt{-\lambda}\ln(r)) & \text{if } \lambda < 0, \\ C_1 \ln(r) + C_2 & \text{if } \lambda = 0, \\ C_1 r^{\sqrt{\lambda}} + C_2 r^{-\sqrt{\lambda}} & \text{if } \lambda > 0, \end{cases}$$

$$\Theta(\theta) = \begin{cases} D_1 e^{\sqrt{-\lambda}\theta} + D_2 e^{-\sqrt{-\lambda}\theta} & \text{if } \lambda < 0, \\ D_1\theta + D_2 & \text{if } \lambda = 0, \\ D_1 \cos(\sqrt{\lambda}\theta) + D_2 \sin(\sqrt{\lambda}\theta) & \text{if } \lambda > 0, \end{cases}$$

where C_1, C_2, D_1, D_2 are constants. Now, according to page 196 of the textbook, the equation for Θ holds at the interval $(0, 2\pi)$. In order for $\Theta(\theta)$ to be twice differentiable (so that $\frac{d^2\Theta}{d\theta^2}$ makes sense, after all) for all $\theta \in \mathbb{R}$, we need to impose the periodic boundary conditions

$$\begin{aligned} \Theta(0) &= \Theta(2\pi), \\ \frac{d}{d\theta}\Theta(0) &= \frac{d}{d\theta}\Theta(2\pi), \end{aligned}$$

which will impose constraints on the constants D_1, D_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then we have

$$\begin{aligned} &\Theta(\theta) = D_1 e^{\sqrt{-\lambda}\theta} + D_2 e^{-\sqrt{-\lambda}\theta}, \\ &\Theta(0) = \Theta(2\pi), \end{aligned}$$

which implies $D_1 + D_2 = D_1 e^{2\pi\sqrt{-\lambda}} + D_2 e^{-2\pi\sqrt{-\lambda}}$. We also have

$$\begin{aligned} \frac{d}{d\theta}\Theta(\theta) &= \sqrt{-\lambda}(D_1 e^{\sqrt{-\lambda}x} - D_2 e^{-\sqrt{-\lambda}x}), \\ \frac{d}{d\theta}\Theta(0) &= \frac{d}{d\theta}\Theta(2\pi), \end{aligned}$$

which implies $D_1 - D_2 = D_1 e^{2\pi\sqrt{-\lambda}} - D_2 e^{-2\pi\sqrt{-\lambda}}$. Now we will solve for the constants D_1, D_2. We have formulated the linear system

$$\begin{aligned} D_1 + D_2 &= D_1 e^{2\pi\sqrt{-\lambda}} + D_2 e^{-2\pi\sqrt{-\lambda}}, \\ D_1 - D_2 &= D_1 e^{2\pi\sqrt{-\lambda}} - D_2 e^{-2\pi\sqrt{-\lambda}}, \end{aligned}$$

and we can algebraically rearrange each equation in the system to write

$$\begin{aligned} D_1(1 - e^{2\pi\sqrt{-\lambda}}) &= -D_2(1 - e^{-2\pi\sqrt{-\lambda}}), \\ D_1(1 - e^{2\pi\sqrt{-\lambda}}) &= D_2(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

We can combine the two equations in the system to deduce

$$\begin{aligned} D_1(1 - e^{2\pi\sqrt{-\lambda}}) &= -D_2(1 - e^{-2\pi\sqrt{-\lambda}}) \\ &= -D_1(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

Since we are currently in the case $\lambda < 0$, we have $1 - e^{2\pi\sqrt{-\lambda}} \neq 0$, and so we can divide $1 - e^{2\pi\sqrt{-\lambda}}$ from both sides of our previous equation to conclude $C_1 = -C_1$, or $C_1 = 0$. Likewise, we can combine the two equations in the system to deduce

$$\begin{aligned} D_2(1 - e^{-2\pi\sqrt{-\lambda}}) &= D_1(1 - e^{2\pi\sqrt{-\lambda}}) \\ &= -D_2(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

Since we are currently in the case $\lambda < 0$, we have $1 - e^{-2\pi\sqrt{-\lambda}} \neq 0$, and so we can divide $1 - e^{-2\pi\sqrt{-\lambda}}$ from both sides of our previous equation to conclude $D_2 = -D_2$, or $D_2 = 0$. So we have

$$\begin{aligned} \Theta(\theta) &= C_1 e^{\sqrt{-\lambda}\theta} + C_2 e^{-\sqrt{-\lambda}\theta} \\ &= 0e^{\sqrt{-\lambda}\theta} + 0e^{-\sqrt{-\lambda}\theta} \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} w(r,\theta) &= R(r)\Theta(\theta) \\ &= R(r) \cdot 0 \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 2: Suppose $\lambda = 0$. Then we have

$$\begin{aligned} \Theta(\theta) &= D_1\theta + D_2, \\ \Theta(0) &= \Theta(2\pi), \end{aligned}$$

which implies $D_1 = 0$, and so we have

$$\begin{aligned} \Theta(\theta) &= D_1\theta + D_2 \\ &= D_1 \cdot 0 + D_2 \\ &= D_2. \end{aligned}$$

The derivative is

$$\begin{aligned} \frac{d}{d\theta}\Theta(\theta) &= \frac{d}{d\theta}(D_2) \\ &= 0, \end{aligned}$$

which clearly satisfies $\frac{d}{d\theta}\Theta(0) = 0 = \frac{d}{d\theta}\Theta(2\pi)$. Therefore, if we write $\frac{\alpha_0}{2} = C_2 D_2$, then we have

$$\begin{aligned} w_0(r,\theta) &= R(r)\Theta(\theta) \\ &= (C_1 \ln(r) + C_2)D_2 \\ &= C_1 D_2 \ln(r) + C_2 D_2 \\ &= \frac{\alpha_0}{2}\ln(r) + \frac{\beta_0}{2}, \end{aligned}$$

which is a nontrivial smooth solution that is also bounded in D.

- Case 3: Suppose $\lambda > 0$. Then we have

$$\begin{aligned}\Theta(\theta) &= D_1\cos(\sqrt{\lambda}\theta) + D_2\sin(\sqrt{\lambda}\theta),\\ \Theta(0) &= \Theta(2\pi),\end{aligned}$$

which implies

$$D_1 = D_1\cos(2\pi\sqrt{\lambda}) + D_2\sin(2\pi\sqrt{\lambda}). \tag{1}$$

We also have

$$\begin{aligned}\frac{d}{d\theta}\Theta(\theta) &= \sqrt{\lambda}(-D_1\sin(\sqrt{\lambda}\theta) + D_2\cos(\sqrt{\lambda}\theta)),\\ \frac{d}{d\theta}\Theta(0) &= \frac{d}{d\theta}\Theta(2\pi),\end{aligned}$$

which implies

$$D_2 = -D_1\sin(2\pi\sqrt{\lambda}) + D_2\cos(2\pi\sqrt{\lambda}). \tag{2}$$

Now, we claim that, if either $\sin(2\pi\sqrt{\lambda}) \neq 0$ or $\cos(2\pi\sqrt{\lambda}) \neq 1$, then we have $D_1 = 0$ and $D_2 = 0$.

  - Subcase 1: Suppose $\sin(2\pi\sqrt{\lambda}) \neq 0$. Multiply both sides of (1) by $-\cos(2\pi\sqrt{\lambda})$ and both sides of (2) by $\sin(2\pi\sqrt{\lambda})$ to obtain

$$\begin{aligned}-D_1\cos(2\pi\sqrt{\lambda}) &= -D_1\cos^2(2\pi\sqrt{\lambda}) - D_2\sin(2\pi\sqrt{\lambda})\cos(2\pi\sqrt{\lambda}),\\ D_2\sin(2\pi\sqrt{\lambda}) &= -D_1\sin^2(2\pi\sqrt{\lambda}) + D_2\cos(2\pi\sqrt{\lambda})\sin(2\pi\sqrt{\lambda}),\end{aligned}$$

  from which we can add up both sides of the two equations to get

$$-D_1\cos(2\pi\sqrt{\lambda}) + D_2\sin(2\pi\sqrt{\lambda}) = -D_1. \tag{3}$$

  We equate (1) and (3) to get

$$\cancel{D_1\cos(2\pi\sqrt{\lambda})} - D_2\sin(2\pi\sqrt{\lambda}) = \cancel{D_1\cos(2\pi\sqrt{\lambda})} + D_2\sin(2\pi\sqrt{\lambda}),$$

  which simplifies to

$$-D_2\cancel{\sin(2\pi\sqrt{\lambda})} = D_2\cancel{\sin(2\pi\sqrt{\lambda})}.$$

  Since we assumed $\sin(2\pi\sqrt{\lambda}) \neq 0$, we can divide both sides by $\sin(2\pi\sqrt{\lambda})$ to get $-D_2 = D_2$, which means $D_2 = 0$. Substitute $D_2 = 0$ into (2) to obtain

$$0 = -D_1\sin(2\pi\sqrt{\lambda}),$$

  which implies $D_1 = 0$ because, once again, we assumed $\sin(2\pi\sqrt{\lambda}) \neq 0$.

  - Subcase 2: Suppose $\cos(2\pi\sqrt{\lambda}) \neq 1$. Then we can rewrite (1) and (2) as

$$D_1(1 - \cos(2\pi\sqrt{\lambda})) = D_2\sin(2\pi\sqrt{\lambda}), \tag{4}$$

$$D_2(1 - \cos(2\pi\sqrt{\lambda})) = -D_1\sin(2\pi\sqrt{\lambda}),. \tag{5}$$

  Multiply both sides of (4) by D_1 and both sides of (5) by D_2 to obtain

$$\begin{aligned}D_1^2(1 - \cos(2\pi\sqrt{\lambda})) &= D_1D_2\sin(2\pi\sqrt{\lambda}),\\ D_2^2(1 - \cos(2\pi\sqrt{\lambda})) &= -D_1D_2\sin(2\pi\sqrt{\lambda}),\end{aligned}$$

  from which we can add up both sides of the two equations to get

$$(D_1^2 + D_2^2)(1 - \cos(2\pi\sqrt{\lambda})) = 0.$$

  Since we assumed $\cos(2\pi\sqrt{\lambda}) \neq 1$, we must conclude $D_1^2 + D_2^2 = 0$, which forces $D_1 = 0$ and $D_2 = 0$.

So we have proved our claim. Now that we have established our claim, we would have

$$\begin{aligned}\Theta(\theta) &= D_1\cos(\sqrt{\lambda}\theta) + D_2\sin(\sqrt{\lambda}\theta)\\ &= 0\cos(\sqrt{\lambda}\theta) + 0\sin(\sqrt{\lambda}\theta)\\ &= 0,\end{aligned}$$

which would imply that $w(r,\theta) = R(r)\Theta(\theta)$ is a trivial solution. Therefore, to find a nontrivial solution for this case, we should assume both

$$\begin{aligned}\sin(2\pi\sqrt{\lambda}) &= 0,\\ 1 - \cos(2\pi\sqrt{\lambda}) &= 0,\end{aligned}$$

which imply $2\pi\sqrt{\lambda} = 2n\pi$, or equivalently

$$\lambda_n = \lambda = n^2,$$

and so we have

$$\begin{aligned}\Theta_n(\theta) &= D_{1,n}\cos(\sqrt{\lambda_n}\theta) + D_{2,n}\sin(\sqrt{\lambda_n}\theta)\\ &= D_{1,n}\cos(\sqrt{n^2}\theta) + D_{2,n}\sin(\sqrt{n^2}\theta)\\ &= D_{1,n}\cos(n\theta) + D_{2,n}\sin(n\theta)\end{aligned}$$

and

$$\begin{aligned}R_n(r) &= C_{1,n}r^{\sqrt{\lambda_n}} + C_{2,n}r^{-\sqrt{\lambda_n}}\\ &= C_{1,n}r^{\sqrt{n^2}} + C_{2,n}r^{-\sqrt{n^2}}\\ &= C_{1,n}r^n + C_{2,n}r^{-n}\end{aligned}$$

for $n = 1, 2, 3, \ldots$. Observe that r^n for $n = 1, 2, \ldots$ is unbounded as $r \to \infty$. Following page 197 of the textbook, we only consider bounded solutions and disregard any solutions that are unbounded as $r \to \infty$, and so we shall impose the condition $C_{1,n} = 0$. So we have

$$\begin{aligned}R_n(r) &= C_{1,n}r^n + C_{2,n}r^{-n}\\ &= 0r^n + C_{2,n}r^{-n}\\ &= C_{2,n}r^{-n}.\end{aligned}$$

Therefore, if we write $\alpha_n := C_{2,n}D_{1,n}$ and $\beta_n := C_{2,n}D_{2,n}$, then we have

$$\begin{aligned}w_n(r,\theta) &= R_n(r)\Theta_n(\theta)\\ &= (C_{2,n}r^{-n})(D_{1,n}\cos(n\theta) + D_{2,n}\sin(n\theta))\\ &= r^{-n}(C_{2,n}D_{1,n}\cos(n\theta) + C_{2,n}D_{2,n}\sin(n\theta))\\ &= r^{-n}(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta)).\end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial smooth solution that is also bounded in D because we also have the assumption $\lim_{r\to\infty} w(r,\theta) = 0$.

We recall that an addition of smooth solutions is again a smooth solution. So that means, as we have established already that each $w_n(r,\theta)$ is a nontrivial smooth solution for $n = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned}w(r,\theta) &= w_0(r,\theta) + \sum_{n=1}^{\infty} w_n(r,\theta)\\ &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^{-n}(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta))\end{aligned}$$

is the general solution of the Laplace equation that is bounded in D. Next, we will compute the Fourier coefficients $\alpha_0, \alpha_n, \beta_n$. We have

$$\begin{aligned}w(2,\theta) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} 2^{-n}(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta))\\ &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} 2^{-n}\alpha_n\cos(n\theta) + \sum_{n=1}^{\infty} 2^{-n}\beta_n\sin(n\theta)\end{aligned}$$

and the given boundary condition

$$w(2,\theta) = 2\sin(\theta).$$

Both our expressions of $w(2,\theta)$ yield

$$\frac{\alpha_0}{2} + \sum_{n=1}^{\infty} 2^{-n}\alpha_n\cos(n\theta) + \sum_{n=1}^{\infty} 2^{-n}\beta_n\sin(n\theta) = 2\sin(\theta).$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to find the Fourier coefficients

$$\alpha_n = 0$$

for $n = 0, 1, 2, \ldots$ and

$$\begin{aligned}\beta_1 &= 4,\\ \beta_n &= 0\end{aligned}$$

for $n = 2, 3, 4, \ldots$. Therefore, our formal solution in polar coordinates is

$$\begin{aligned} w(r,\theta) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^{-n}(\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)) \\ &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} \beta_n r^n \sin(n\theta) \\ &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n r^{-n} \cos(n\theta) + \left(\beta_1 r^{-1} \sin(1\theta) + \sum_{n=2,3,4,\ldots} \beta_n r^{-n} \sin(n\theta)\right) \\ &= \frac{0}{2} + \sum_{n=1}^{\infty} 0 r^n \cos(n\theta) + \left(4r^{-1} \sin(1\theta) + \sum_{n=2,3,4,\ldots} 0 r^{-n} \sin(n\theta)\right) \\ &= \boxed{\frac{4}{r}\sin(\theta)}. \end{aligned}$$

In Cartesian coordinates, our formal solution is

$$\begin{aligned} u(x,y) &= u(x(r,\theta), y(r,\theta)) \\ &= w(r,\theta) \\ &= \frac{4}{r}\sin(\theta) \\ &= \frac{4r\sin(\theta)}{r^2} \\ &= \boxed{\frac{4y}{x^2+y^2}}, \end{aligned}$$

where we used $y = r\sin(\theta)$ and $r^2 = x^2 + y^2$. □

7.14. Consider the domain $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$ and the Neumann problem

$$\begin{aligned} \Delta u &= 0 && (x,y) \in D, \\ \frac{\partial u}{\partial n} &= \alpha x^2 + \beta y + \gamma && (x,y) \in \partial D, \end{aligned}$$

where α, β, γ are real constants.

(a) Find the values of α, β, γ for which the problem is not solvable.

Solution. Lemma 7.4 of the textbook states that a necessary condition for the existence of a solution to the Neumann problem

$$\begin{aligned} \Delta u &= f(x,y) && (x,y) \in D, \\ \frac{\partial u}{\partial n} &= g(x,y) && (x,y) \in \partial D, \end{aligned}$$

is

$$\int_{\partial D} g(x(s), y(s))\, ds = \iint_D f(x,y)\, dx\, dy,$$

where $(x(s), y(s))$ is a parameterization of ∂D. For this specific exercise, we have $f(x,y) = 0$ and $g(x,y) = \alpha x^2 + \beta y + \gamma$. So the necessary condition becomes

$$\int_{\partial D} \alpha(x(s))^2 + \beta y(s) + \gamma\, ds = \iint_D 0\, dx\, dy.$$

Now, the domain $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$ implies the specific parameterizations

$$\begin{aligned} x(s) &= 2\cos(\theta), \\ y(s) &= 2\sin(\theta) \end{aligned}$$

along the boundary ∂D. The necessary condition therefore becomes

$$\int_0^{2\pi} \alpha(2\cos(\theta))^2 + \beta(2\sin(\theta)) + \gamma\, d\theta = \iint_D 0\, dx\, dy,$$

or equivalently

$$\int_0^{2\pi} 4\alpha\cos^2(\theta) + 2\beta\sin(\theta) + \gamma\, d\theta = 0.$$

But we can also rewrite the left hand side as

$$\begin{aligned}\int_0^{2\pi} 4\alpha\cos^2(\theta) + 2\beta\sin(\theta) + \gamma\, d\theta &= 4\alpha^2\int_0^{2\pi}\cos^2(\theta)\,d\theta + 2\beta\int_0^{2\pi}\sin(\theta)\,d\theta + \gamma\int_0^{2\pi} 1\,d\theta\\ &= 4\alpha^2\cdot\pi + 2\beta\cdot 0 + \gamma\cdot 2\pi\\ &= 2\pi(2\alpha^2+\gamma).\end{aligned}$$

So the necessary condition finally becomes

$$2\pi(2\alpha^2+\gamma) = 0,$$

which implies

$$2\alpha^2+\gamma = 0,$$

or $\gamma = -2\alpha^2$. We conclude that, if the solution to the Neumann problem exists, then we must have $\gamma = -2\alpha^2$. In other words, this problem is not solvable if we have $\gamma \neq -2\alpha^2$. □

7.15. Let $D = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < \pi, 0 < y < \pi\}$. Denote its boundary by ∂D.

(a) Assume $v_{xx} + v_{yy} + xv_x + yv_y > 0$ in D. Prove that v has no local maximum in D.

Solution. We will prove by contradiction. Suppose instead that v has a local maximum at some $(x_0, y_0) \in D$. Then all the first partial derivatives of v at (x_0, y_0) are zero and the second partial derivatives are nonpositive (negative or zero); that is, we have

$$\begin{aligned}v_x(x_0,y_0) &= 0,\\ v_y(x_0,y_0) &= 0,\\ v_{xx}(x_0,y_0) &\le 0,\\ v_{yy}(x_0,y_0) &\le 0.\end{aligned}$$

So, at $(x_0, y_0) \in D$, we have

$$\begin{aligned}v_{xx}+v_{yy}+xv_x+yv_y &= v_{xx}+v_{yy}+x\cdot 0+y\cdot 0\\ &= v_{xx}+v_{yy}\\ &\le 0+0\\ &= 0,\end{aligned}$$

which contradicts the assumption $v_{xx} + v_{yy} + xv_x + yv_y > 0$ in D. We conclude that v has no local maximum in D. □

(b) Consider the problem

$$\begin{aligned}u_{xx}+u_{yy}+xu_x+yu_y &= 0 && \text{if } (x,y)\in D,\\ u(x,y) &= f(x,y) && \text{if } (x,y)\in\partial D,\end{aligned}$$

where f is a continuous function. Show that, if u is a solution, then the maximum of u is achieved on the boundary ∂D. *Hint:* Use the auxiliary function $v_\epsilon(x, y) = u(x, y) + \epsilon x^2$ for any $\epsilon > 0$.

Solution. Following the given hint, define $v_\epsilon(x, y) = u(x, y) + \epsilon x^2$ for any $\epsilon > 0$. Then we have the first and second partial derivatives

$$\begin{aligned}(v_\epsilon)_x(x,y) &= u_x(x,y)+2\epsilon x,\\ (v_\epsilon)_y(x,y) &= u_y(x,y),\\ (v_\epsilon)_{xx}(x,y) &= u_{xx}(x,y)+2\epsilon,\\ (v_\epsilon)_{yy}(x,y) &= u_{yy}(x,y).\end{aligned}$$

So we have

$$\begin{aligned}(v_\epsilon)_{xx}+(v_\epsilon)_{yy}+x(v_\epsilon)_x+y(v_\epsilon)_y &= (u_{xx}+2\epsilon)+u_{yy}+x(u_x+2\epsilon x)+yu_y\\ &= u_{xx}+u_{yy}+xu_x+yu_y+2\epsilon(1+x^2)\\ &= 0+2\epsilon(1+x^2)\\ &= 2\epsilon(1+x^2)\\ &> 2\epsilon(1+0^2)\\ &= 2\epsilon\\ &> 0\end{aligned}$$

in D. Note that we have just obtained exactly the partial differential equation described in part (a). Since f is continuous on ∂D and $u = f$ on ∂D, it follows that u is continuous on ∂D and smooth in D. Consequently, $v_\epsilon = u + \epsilon x^2$ is also continuous on ∂D and smooth in D. This rules out the possibility that v_ϵ does not have a maximum on either D or ∂D; in other words, $\max_{D\cup\partial D} v_\epsilon$ exists. But part (a) asserts that v_ϵ has no local maximum in D. So we conclude that the only place on which the maximum of v_ϵ exists is ∂D. This implies that, for all $\epsilon > 0$, we have

$$\begin{aligned}\max_{D\cup\partial D} v_\epsilon &= \max_{\partial D} v_\epsilon \\ &= \max_{\partial D}(u + \epsilon x^2) \\ &= \max_{\partial D} u + \epsilon(\max_{\partial D} x^2) \\ &= \max_{\partial D} u + \epsilon\pi^2.\end{aligned}$$

In other words, we have

$$v_\epsilon(x, y) \le \max_{\partial D} u(x, y) + \epsilon\pi^2$$

for all $(x, y) \in D$. Finally, we can send $\epsilon \to 0^+$ both sides of our latest equation, writing

$$\lim_{\epsilon\to 0^+} v_\epsilon(x, y) \le \lim_{\epsilon\to 0^+} (\max_{\partial D} u(x, y) + \epsilon\pi^2),$$

to conclude

$$u(x, y) \le \max_{\partial D} u(x, y)$$

for all $(x, y) \in D$. In other words, we conclude that the maximum of u is achieved on ∂D. □

(c) Show that the problem formulated in part (b) has at most one solution.

Solution. Suppose $u_1(x, y)$ and $u_2(x, y)$ are two solutions of the problem formulated in part (b). First, define $w(x, y) = u_1(x, y) - u_2(x, y)$. Then w and $-w$ solve

$$\begin{aligned} w_{xx} + w_{yy} + xw_x + yw_y &= 0 \quad \text{if } (x, y) \in D, \\ w(x, y) &= 0 \quad \text{if } (x, y) \in \partial D.\end{aligned}$$

By the Weak Maximum Principle, we have

$$\begin{aligned}\max_D w(x, y) &\le \max_{\partial D} w(x, y), \\ \max_D -w(x, y) &\le \max_{\partial D} -w(x, y),\end{aligned}$$

which implies

$$\begin{aligned} w(x, y) &\le \max_{\partial D} w(x, y) = \max_{\partial D} 0 = 0, \\ -w(x, y) &\le \max_{\partial D} -w(x, y) = \max_{\partial D} 0 = 0\end{aligned}$$

for all $(x, y) \in D$. Note that $-w(x, y) \le 0$ is equivalent to $w(x, y) \ge 0$. So we conclude

$$0 \le w(x, y) \le 0,$$

which forces $w = 0$, or $u_1 - u_2 = 0$ in D. In other words, we have $u_1 = u_2$ in D, meaning that the problem in part (b) has at most one solution in D. It also goes without saying that the problem in part (b) also has at most one solution in ∂D because we have been dealing with $w = 0$ on ∂D. □

7.20. Consider the domain $D = \{(r, \theta) \in \mathbb{R} \times [0, 2\pi] \mid 2 < r < 4, 0 \le \theta \le 2\pi\}$. Find $u(r, \theta)$ that solves

$$\begin{aligned} \Delta u &= 0 \qquad\qquad 2 < r < 4, 0 \le \theta \le 2\pi, \\ u(2, \theta) = 0, u(4, \theta) &= \sin(\theta), \quad 0 \le \theta \le 2\pi.\end{aligned}$$

Solution. We want to find a solution of the form

$$u(r, \theta) = R(r)\Theta(\theta).$$

Our partial derivatives are

$$\begin{aligned} u_r(r, \theta) &= R_r(r)\Theta(\theta), \\ u_{rr}(r, \theta) &= R_{rr}(r)\Theta(\theta), \\ u_{\theta\theta}(r, \theta) &= R(r)\Theta_{\theta\theta}(\theta).\end{aligned}$$

So the partial differential equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \Delta w = 0$$

becomes

$$R_{rr}(r)\Theta(\theta) + \frac{1}{r}R_r(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta_{\theta\theta}(\theta) = 0,$$

which we can algebraically rearrange to write

$$-\frac{r^2R_{rr}(r) + rR_r(r)}{R(r)} = \frac{\Theta_{\theta\theta}(\theta)}{\Theta(\theta)} = -\lambda,$$

where λ is a constant in both r and θ. This produces the system of two ordinary differential equations

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda R = 0$$
$$\frac{d^2\Theta}{dr^2} + \lambda\Theta = 0.$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$R(r) = \begin{cases} C_1\cos(\sqrt{-\lambda}\ln(r)) + C_2\sin(\sqrt{-\lambda}\ln(r)) & \text{if } \lambda < 0, \\ C_1\ln(r) + C_2 & \text{if } \lambda = 0, \\ C_1r^{\sqrt{\lambda}} + C_2r^{-\sqrt{\lambda}} & \text{if } \lambda > 0, \end{cases}$$

$$\Theta(\theta) = \begin{cases} D_1e^{\sqrt{-\lambda}\theta} + D_2e^{-\sqrt{-\lambda}\theta} & \text{if } \lambda < 0, \\ D_1\theta + D_2 & \text{if } \lambda = 0, \\ D_1\cos(\sqrt{\lambda}\theta) + D_2\sin(\sqrt{\lambda}\theta) & \text{if } \lambda > 0, \end{cases}$$

where C_1, C_2, D_1, D_2 are constants. Now, according to page 196 of the textbook, the equation for Θ holds at the interval $(0, 2\pi)$. In order for $\Theta(\theta)$ to be twice differentiable (so that $\frac{d^2\Theta}{d\theta^2}$ makes sense, after all) for all $\theta \in \mathbb{R}$, we need to impose the periodic boundary conditions

$$\Theta(0) = \Theta(2\pi),$$
$$\frac{d}{d\theta}\Theta(0) = \frac{d}{d\theta}\Theta(2\pi),$$

which will impose constraints on the constants D_1, D_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then we have

$$\Theta(\theta) = D_1e^{\sqrt{-\lambda}\theta} + D_2e^{-\sqrt{-\lambda}\theta},$$
$$\Theta(0) = \Theta(2\pi),$$

which implies $D_1 + D_2 = D_1e^{2\pi\sqrt{-\lambda}} + D_2e^{-2\pi\sqrt{-\lambda}}$. We also have

$$\frac{d}{d\theta}\Theta(\theta) = \sqrt{-\lambda}(D_1e^{\sqrt{-\lambda}x} - D_2e^{-\sqrt{-\lambda}x}),$$
$$\frac{d}{d\theta}\Theta(0) = \frac{d}{d\theta}\Theta(2\pi),$$

which implies $D_1 - D_2 = D_1e^{2\pi\sqrt{-\lambda}} - D_2e^{-2\pi\sqrt{-\lambda}}$. Now we will solve for the constants D_1, D_2. We have formulated the linear system

$$D_1 + D_2 = D_1e^{2\pi\sqrt{-\lambda}} + D_2e^{-2\pi\sqrt{-\lambda}},$$
$$D_1 - D_2 = D_1e^{2\pi\sqrt{-\lambda}} - D_2e^{-2\pi\sqrt{-\lambda}},$$

and we can algebraically rearrange each equation in the system to write

$$D_1(1 - e^{2\pi\sqrt{-\lambda}}) = -D_2(1 - e^{-2\pi\sqrt{-\lambda}}),$$
$$D_1(1 - e^{2\pi\sqrt{-\lambda}}) = D_2(1 - e^{-2\pi\sqrt{-\lambda}}).$$

We can combine the two equations in the system to deduce

$$D_1(1 - e^{2\pi\sqrt{-\lambda}}) = -D_2(1 - e^{-2\pi\sqrt{-\lambda}})$$
$$= -D_1(1 - e^{-2\pi\sqrt{-\lambda}}).$$

Since we are currently in the case $\lambda < 0$, we have $1 - e^{2\pi\sqrt{-\lambda}} \neq 0$, and so we can divide $1 - e^{2\pi\sqrt{-\lambda}}$ from both sides of our previous equation to conclude $C_1 = -C_1$, or $C_1 = 0$. Likewise, we can combine the two equations in the system to deduce

$$\begin{aligned} D_2(1 - e^{-2\pi\sqrt{-\lambda}}) &= D_1(1 - e^{2\pi\sqrt{-\lambda}}) \\ &= -D_2(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

Since we are currently in the case $\lambda < 0$, we have $1 - e^{-2\pi\sqrt{-\lambda}} \neq 0$, and so we can divide $1 - e^{-2\pi\sqrt{-\lambda}}$ from both sides of our previous equation to conclude $D_2 = -D_2$, or $D_2 = 0$. So we have

$$\begin{aligned} \Theta(\theta) &= C_1 e^{\sqrt{-\lambda}\theta} + C_2 e^{-\sqrt{-\lambda}\theta} \\ &= 0e^{\sqrt{-\lambda}\theta} + 0e^{-\sqrt{-\lambda}\theta} \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} w(r,\theta) &= R(r)\Theta(\theta) \\ &= R(r) \cdot 0 \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 2: Suppose $\lambda = 0$. Then we have

$$\begin{aligned} &\Theta(\theta) = D_1\theta + D_2, \\ &\Theta(0) = \Theta(2\pi), \end{aligned}$$

which implies $D_1 = 0$, and so we have

$$\begin{aligned} \Theta(\theta) &= D_1\theta + D_2 \\ &= D_1 \cdot 0 + D_2 \\ &= D_2. \end{aligned}$$

The derivative is

$$\begin{aligned} \frac{d}{d\theta}\Theta(\theta) &= \frac{d}{d\theta}(D_2) \\ &= 0, \end{aligned}$$

which clearly satisfies $\frac{d}{d\theta}\Theta(0) = 0 = \frac{d}{d\theta}\Theta(2\pi)$. Therefore, if we write $\frac{\alpha_0}{2} = C_1D_2$ and $\frac{\beta_0}{2} = C_2D_2$, then we have

$$\begin{aligned} u_0(r,\theta) &= R(r)\Theta(\theta) \\ &= (C_1\ln(r) + C_2)D_2 \\ &= C_1D_2\ln(r) + C_2D_2 \\ &= \frac{\alpha_0}{2}\ln(r) + \frac{\beta_0}{2}, \end{aligned}$$

which is a nontrivial smooth solution that is also bounded in D.

- Case 3: Suppose $\lambda > 0$. Then we have

$$\begin{aligned} &\Theta(\theta) = D_1\cos(\sqrt{\lambda}\theta) + D_2\sin(\sqrt{\lambda}\theta), \\ &\Theta(0) = \Theta(2\pi), \end{aligned}$$

which implies

$$D_1 = D_1\cos(2\pi\sqrt{\lambda}) + D_2\sin(2\pi\sqrt{\lambda}). \tag{1}$$

We also have

$$\begin{aligned} &\frac{d}{d\theta}\Theta(\theta) = \sqrt{\lambda}(-D_1\sin(\sqrt{\lambda}\theta) + D_2\cos(\sqrt{\lambda}\theta)), \\ &\frac{d}{d\theta}\Theta(0) = \frac{d}{d\theta}\Theta(2\pi), \end{aligned}$$

which implies

$$D_2 = -D_1\sin(2\pi\sqrt{\lambda}) + D_2\cos(2\pi\sqrt{\lambda}). \tag{2}$$

Now, we claim that, if either $\sin(2\pi\sqrt{\lambda}) \neq 0$ or $\cos(2\pi\sqrt{\lambda}) \neq 1$, then we have $D_1 = 0$ and $D_2 = 0$.

- Subcase 1: Suppose $\sin(2\pi\sqrt{\lambda}) \neq 0$. Multiply both sides of (1) by $-\cos(2\pi\sqrt{\lambda})$ and both sides of (2) by $\sin(2\pi\sqrt{\lambda})$ to obtain

$$\begin{aligned} -D_1\cos(2\pi\sqrt{\lambda}) &= -D_1\cos^2(2\pi\sqrt{\lambda}) - D_2\sin(2\pi\sqrt{\lambda})\cos(2\pi\sqrt{\lambda}), \\ D_2\sin(2\pi\sqrt{\lambda}) &= -D_1\sin^2(2\pi\sqrt{\lambda}) + D_2\cos(2\pi\sqrt{\lambda})\sin(2\pi\sqrt{\lambda}), \end{aligned}$$

from which we can add up both sides of the two equations to get

$$-D_1\cos(2\pi\sqrt{\lambda}) + D_2\sin(2\pi\sqrt{\lambda}) = -D_1. \tag{3}$$

We equate (1) and (3) to get

$$\cancel{D_1\cos(2\pi\sqrt{\lambda})} - D_2\sin(2\pi\sqrt{\lambda}) = \cancel{D_1\cos(2\pi\sqrt{\lambda})} + D_2\sin(2\pi\sqrt{\lambda}),$$

which simplifies to

$$-D_2\cancel{\sin(2\pi\sqrt{\lambda})} = D_2\cancel{\sin(2\pi\sqrt{\lambda})}.$$

Since we assumed $\sin(2\pi\sqrt{\lambda}) \neq 0$, we can divide both sides by $\sin(2\pi\sqrt{\lambda})$ to get $-D_2 = D_2$, which means $D_2 = 0$. Substitute $D_2 = 0$ into (2) to obtain

$$0 = -D_1\sin(2\pi\sqrt{\lambda}),$$

which implies $D_1 = 0$ because, once again, we assumed $\sin(2\pi\sqrt{\lambda}) \neq 0$.
- Subcase 2: Suppose $\cos(2\pi\sqrt{\lambda}) \neq 1$. Then we can rewrite (1) and (2) as

$$D_1(1 - \cos(2\pi\sqrt{\lambda})) = D_2\sin(2\pi\sqrt{\lambda}), \tag{4}$$
$$D_2(1 - \cos(2\pi\sqrt{\lambda})) = -D_1\sin(2\pi\sqrt{\lambda}),. \tag{5}$$

Multiply both sides of (4) by D_1 and both sides of (5) by D_2 to obtain

$$\begin{aligned} D_1^2(1 - \cos(2\pi\sqrt{\lambda})) &= D_1D_2\sin(2\pi\sqrt{\lambda}), \\ D_2^2(1 - \cos(2\pi\sqrt{\lambda})) &= -D_1D_2\sin(2\pi\sqrt{\lambda}), \end{aligned}$$

from which we can add up both sides of the two equations to get

$$(D_1^2 + D_2^2)(1 - \cos(2\pi\sqrt{\lambda})) = 0.$$

Since we assumed $\cos(2\pi\sqrt{\lambda}) \neq 1$, we must conclude $D_1^2 + D_2^2 = 0$, which forces $D_1 = 0$ and $D_2 = 0$.

So we have proved our claim. Now that we have established our claim, we would have

$$\begin{aligned} \Theta(\theta) &= D_1\cos(\sqrt{\lambda}\theta) + D_2\sin(\sqrt{\lambda}\theta) \\ &= 0\cos(\sqrt{\lambda}\theta) + 0\sin(\sqrt{\lambda}\theta) \\ &= 0, \end{aligned}$$

which would imply that $w(r,\theta) = R(r)\Theta(\theta)$ is a trivial solution. Therefore, to find a nontrivial solution for this case, we should assume both

$$\begin{aligned} \sin(2\pi\sqrt{\lambda}) &= 0, \\ 1 - \cos(2\pi\sqrt{\lambda}) &= 0, \end{aligned}$$

which imply $2\pi\sqrt{\lambda} = 2n\pi$, or equivalently

$$\lambda_n = \lambda = n^2,$$

and so we have

$$\begin{aligned} \Theta_n(\theta) &= D_{1,n}\cos(\sqrt{\lambda_n}\theta) + D_{2,n}\sin(\sqrt{\lambda_n}\theta) \\ &= D_{1,n}\cos(\sqrt{n^2}\theta) + D_{2,n}\sin(\sqrt{n^2}\theta) \\ &= D_{1,n}\cos(n\theta) + D_{2,n}\sin(n\theta) \end{aligned}$$

and

$$\begin{aligned} R_n(r) &= C_{1,n}r^{\sqrt{\lambda_n}} + C_{2,n}r^{-\sqrt{\lambda_n}} \\ &= C_{1,n}r^{\sqrt{n^2}} + C_{2,n}r^{-\sqrt{n^2}} \\ &= C_{1,n}r^n + C_{2,n}r^{-n} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Therefore, if we write $\alpha_n := C_{1,n}D_{1,n}$, $\beta_n := C_{1,n}D_{2,n}$, $\gamma_n := C_{2,n}D_{1,n}$, $\delta_n := C_{2,n}D_{2,n}$, then we have

$$\begin{aligned} u_n(r,\theta) &= R_n(r)\Theta_n(\theta) \\ &= (C_{1,n}r^n + C_{2,n}r^{-n})(D_{1,n}\cos(n\theta) + D_{2,n}\sin(n\theta)) \\ &= (C_{1,n}D_{1,n}r^n + C_{2,n}D_{1,n}r^{-n})\cos(n\theta) + (C_{1,n}D_{2,n}r^n + C_{2,n}D_{2,n}r^{-n})\sin(n\theta) \\ &= (\alpha_n r^n + \gamma_n r^{-n})\cos(n\theta) + (\beta_n r^n + \delta_n r^{-n})\sin(n\theta). \end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial smooth solution that is also bounded in D.

We recall that an addition of smooth solutions is again a smooth solution. So that means, as we have established already that each $w_n(r,\theta)$ is a nontrivial smooth solution for $n = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned} u(r,\theta) &= u_0(r,\theta) + \sum_{n=1}^{\infty} u_n(r,\theta) \\ &= \frac{\alpha_0}{2}\ln(r) + \frac{\beta_0}{2} + \sum_{n=1}^{\infty}((\alpha_n r^n + \gamma_n r^{-n})\cos(n\theta) + (\beta_n r^n + \delta_n r^{-n})\sin(n\theta)) \\ &= \frac{\alpha_0}{2}\ln(r) + \frac{\beta_0}{2} + \sum_{n=1}^{\infty}(\alpha_n r^n + \gamma_n r^{-n})\cos(n\theta) + \sum_{n=1}^{\infty}(\beta_n r^n + \delta_n r^{-n})\sin(n\theta) \end{aligned}$$

is the general solution of the Laplace equation that is also bounded in D. Next, we will now compute the Fourier coefficients $\alpha_0, \beta_0, \alpha_n, \beta_n, \gamma_n, \delta_n$. We have

$$\begin{aligned} u(2,\theta) &= \frac{\alpha_0}{2}\ln(2) + \frac{\beta_0}{2} + \sum_{n=1}^{\infty}(\alpha_n 2^n + \gamma_n 2^{-n})\cos(n\theta) + \sum_{n=1}^{\infty}(\beta_n 2^n + \delta_n 2^{-n})\sin(n\theta), \\ u(4,\theta) &= \frac{\alpha_0}{2}\ln(4) + \frac{\beta_0}{2} + \sum_{n=1}^{\infty}(\alpha_n 4^n + \gamma_n 4^{-n})\cos(n\theta) + \sum_{n=1}^{\infty}(\beta_n 4^n + \delta_n 4^{-n})\sin(n\theta) \end{aligned}$$

and the given boundary conditions

$$\begin{aligned} u(2,\theta) &= 0, \\ u(4,\theta) &= \sin(\theta). \end{aligned}$$

Both our expressions of $u(2,\theta)$ and $u(4,\theta)$ yield, respectively,

$$\begin{aligned} \frac{\alpha_0}{2}\ln(2) + \frac{\beta_0}{2} + \sum_{n=1}^{\infty}(\alpha_n 2^n + \gamma_n 2^{-n})\cos(n\theta) + \sum_{n=1}^{\infty}(\beta_n 2^n + \delta_n 2^{-n})\sin(n\theta) &= 0, \\ \frac{\alpha_0}{2}\ln(4) + \frac{\beta_0}{2} + \sum_{n=1}^{\infty}(\alpha_n 4^n + \gamma_n 4^{-n})\cos(n\theta) + \sum_{n=1}^{\infty}(\beta_n 4^n + \delta_n 4^{-n})\sin(n\theta) &= \sin(\theta). \end{aligned}$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of each of our two equations above to find

$$\begin{aligned} \alpha_0 = \beta_0 &= 0, \\ \alpha_n 2^n + \gamma_n 2^{-n} &= 0, \\ \beta_n 2^n + \delta_n 2^{-n} &= 0, \\ \alpha_n 4^n + \gamma_n 4^{-n} &= 0, \\ \beta_1 4^n + \delta_n 4^{-n} &= \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n = 2, 3, 4, \ldots \end{cases} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. In particular, we have obtained two linear systems of equations

$$\begin{aligned} \beta_n 2^n + \delta_n 2^{-n} &= 0, \\ \beta_n 4^n + \delta_n 4^{-n} &= \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n = 2, 3, 4, \ldots \end{cases} \end{aligned}$$

and

$$\begin{aligned} \alpha_n 2^n + \gamma_n 2^{-n} &= 0, \\ \alpha_n 4^n + \gamma_n 4^{-n} &= 0. \end{aligned}$$

for $n = 1, 2, 3, \ldots$, and we can simultaneously solve each one of them to obtain

$$\alpha_n = 0,$$
$$\beta_n = \begin{cases} \frac{1}{3} & \text{if } n = 1, \\ 0 & \text{if } n = 2, 3, 4, \ldots, \end{cases}$$
$$\gamma_n = 0,$$
$$\delta_n = \begin{cases} -\frac{4}{3} & \text{if } n = 1, \\ 0 & \text{if } n = 2, 3, 4, \ldots \end{cases}$$

for $n = 1, 2, 3, \ldots$. Therefore, our formal solution in polar coordinates is

$$\begin{aligned}
u(r, \theta) &= \frac{\alpha_0}{2} \ln(r) + \frac{\beta_0}{2} + \sum_{n=1}^{\infty} (\alpha_n r^n + \gamma_n r^{-n}) \cos(n\theta) + \sum_{n=1}^{\infty} (\beta_n r^n + \delta_n r^{-n}) \sin(n\theta) \\
&= \frac{\alpha_0}{2} \ln(r) + \frac{\beta_0}{2} + \sum_{n=1}^{\infty} \alpha_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} \gamma_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} \beta_n r^n \sin(n\theta) + \sum_{n=1}^{\infty} \delta_n r^{-n} \sin(n\theta) \\
&= \frac{\alpha_0}{2} \ln(r) + \frac{\beta_0}{2} + \sum_{n=1}^{\infty} \alpha_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} \gamma_n r^{-n} \cos(n\theta) \\
&\quad + \left(\beta_1 r^1 \sin(1\theta) + \sum_{n=2}^{\infty} \beta_n r^n \sin(n\theta) \right) + \left(\delta_1 r^{-1} \sin(1\theta) + \sum_{n=2}^{\infty} \delta_n r^{-n} \sin(n\theta) \right) \\
&= \frac{0}{2} \ln(r) + \frac{0}{2} + \sum_{n=1}^{\infty} 0 r^n \cos(n\theta) + \sum_{n=1}^{\infty} 0 r^{-n} \cos(n\theta) \\
&\quad + \left(\frac{1}{3} r^1 \sin(1\theta) + \sum_{n=2}^{\infty} 0 r^n \sin(n\theta) \right) + \left(-\frac{4}{3} r^{-1} \sin(1\theta) + \sum_{n=2}^{\infty} 0 r^{-n} \sin(n\theta) \right) \\
&= \frac{1}{3} r \sin(\theta) - \frac{4}{3r} \sin(\theta) \\
&= \boxed{\frac{r^2 - 4}{3r} \sin(\theta)},
\end{aligned}$$

as desired. □

7.22. Consider the domain $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 36\}$. Let $u(x, y)$ solve

$$\begin{aligned}
\Delta u &= 0 && (x, y) \in D, \\
u(x, y) &= \begin{cases} x & \text{if } x < 0, \\ 0 & \text{if } x \geq 0 \end{cases} && (x, y) \in \partial D.
\end{aligned}$$

(a) Prove that we have $u(x, y) < \min\{x, 0\}$ in D.
Hint: Prove that we have $u(x, y) < x$ and $u(x, y) < 0$ in D.

Solution. Define $v(x, y) := u(x, y) - x$. Then v solves

$$\begin{aligned}
\Delta v &= 0 && (x, y) \in D, \\
v(x, y) &= \begin{cases} 0 & \text{if } x < 0, \\ -x & \text{if } x \geq 0 \end{cases} && (x, y) \in \partial D.
\end{aligned}$$

Notice by construction that u and v satisfy

$$\begin{aligned}
u(x, y) &\leq 0, \\
v(x, y) &\leq 0
\end{aligned}$$

on ∂D. By the Weak Maximum Principle, we have

$$\begin{aligned}
\max_D u(x, y) &= \max_{\partial D} u(x, y), \\
\max_D v(x, y) &= \max_{\partial D} v(x, y).
\end{aligned}$$

Also, we have of course

$$\begin{aligned}
u(x, y) &\leq \max_D u(x, y), \\
v(x, y) &\leq \max_D v(x, y)
\end{aligned}$$

for all $(x, y) \in D$. However, if we have

$$u(x, y) = \max_D u(x, y),$$
$$v(x, y) = \max_D v(x, y)$$

at some $(x, y) \in D$, then the Strong Maximum Principle would assert that u and v are constant in D. By continuity of u and v, we would conclude that u and v are also constant on ∂D, but this contradicts the known non-constant functions

$$u(x, y) = \begin{cases} x & \text{if } x < 0, \\ 0 & \text{if } x \geq 0, \end{cases}$$
$$v(x, y) = \begin{cases} 0 & \text{if } x < 0, \\ -x & \text{if } x \geq 0 \end{cases}$$

on ∂D. Therefore, equality is not possible; in other words, we must conclude

$$u(x, y) < \max_D u(x, y),$$
$$v(x, y) < \max_D v(x, y)$$

Therefore, we have

$$u(x, y) < \max_D u(x, y) = \max_{\partial D} u(x, y) = 0,$$
$$v(x, y) < \max_D v(x, y) = \max_{\partial D} v(x, y) = 0$$

in D. This is equivalent to saying

$$u(x, y) < 0,$$
$$u(x, y) < x$$

in D, which is equivalent to $u(x, y) < \min\{x, 0\}$ in D. □

(b) Evaluate $u(0, 0)$ using the mean value principle.

Solution. By the mean value principle (Theorem 7.7 of the textbook, on page 179) applied to D, we have

$$\begin{aligned} u(0,0) &= \frac{1}{2\pi} \int_0^{2\pi} u(0 + 6\cos(\theta), 0 + 6\sin(\theta))\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(6\cos(\theta), 6\sin(\theta))\, d\theta \\ &= \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} 0\, d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos(\theta)\, d\theta + \int_{\frac{3\pi}{2}}^{2\pi} 0\, d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos(\theta)\, d\theta \\ &= \boxed{-\frac{6}{\pi}}, \end{aligned}$$

as desired. □

(c) Using Poisson's formula, evaluate $u(0, y)$ for $0 \leq y < 6$.

Solution. Note that the boundary function in polar coordinates is

$$h(\theta) = w(6, \theta) = \begin{cases} 0 & \text{if } -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \\ 6\cos(\theta) & \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2}. \end{cases}$$

By the Poisson formula from page 202 of the textbook applied to D, we have

$$\begin{aligned} w(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{36 - r^2}{36 - 12r\cos(\theta - \varphi) + r^2} h(\varphi)\, d\varphi \\ &= \frac{1}{2\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{36 - r^2}{36 - 12r\cos(\theta - \varphi) + r^2} 6\cos(\varphi)\, d\varphi \\ &= \frac{3}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{36 - r^2}{36 - 12r(\cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi)) + r^2} \cos(\varphi)\, d\varphi. \end{aligned}$$

In Cartesian coordinates, this is

$$\begin{aligned}
u(x, y) &= u(x(r,\theta), y(r,\theta)) \\
&= w(r,\theta) \\
&= \frac{3}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{36 - r^2}{36 - 12r(\cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi)) + r^2}\cos(\varphi)\, d\varphi \\
&= \frac{3}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{36 - r^2}{36 - 12(r\cos(\theta)\cos(\varphi) + r\sin(\theta)\sin(\varphi)) + r^2}\cos(\varphi)\, d\varphi \\
&= \frac{3}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{36 - (x^2 + y^2)}{36 - 12(x\cos(\varphi) + y\sin(\varphi)) + x^2 + y^2}\cos(\varphi)\, d\varphi.
\end{aligned}$$

On the line $x = 0$, we obtain

$$\begin{aligned}
u(0, 0) &= \frac{3}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{36 - (0^2 + 0^2)}{36 - 12(0\cos(\varphi) + 0\sin(\varphi)) + 0^2 + 0^2}\cos(\varphi)\, d\varphi \\
&= \frac{3}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(\varphi)\, d\varphi \\
&= \frac{3}{\pi}\cdot -2 \\
&= -\frac{6}{\pi}
\end{aligned}$$

and, if we can employ the substitution $u = 36 - 12y\sin(\varphi) + y^2$, which implies $du = -12y\cos(\varphi)\, d\varphi$, then we have

$$\begin{aligned}
u(0, y) &= \frac{3}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{36 - (0^2 + y^2)}{36 - 12(0\cos(\varphi) + y\sin(\varphi)) + 0^2 + y^2}\cos(\varphi)\, d\varphi \\
&= \frac{3}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{36 - y^2}{36 - 12y\sin(\varphi) + y^2}\cos(\varphi)\, d\varphi \\
&= \frac{3}{\pi}(36 - y^2)\int_{36-12y+y^2}^{36+12y+y^2} \frac{1}{u}\left(-\frac{du}{12y}\right) \\
&= -\frac{1}{4\pi}\frac{36 - y^2}{y}\int_{(6-y)^2}^{(6+y)^2} \frac{1}{u}\, du \\
&= -\frac{1}{2\pi}\frac{36 - y^2}{y}\ln\left(\frac{6 + y}{6 - y}\right)
\end{aligned}$$

for all $0 < y < 6$. In summary, we have

$$u(0, y) = \boxed{\begin{cases} -\frac{6}{\pi} & \text{if } y = 0, \\ -\frac{1}{2\pi}\frac{36-y^2}{y}\ln\left(\frac{6+y}{6-y}\right) & \text{if } 0 < y < 6 \end{cases}}$$

for all $0 \le y < 6$. □

Remark. *By using l'Hôpital's rule, we see that our expression of $u(0,y)$ satisfies*

$$\begin{aligned}
\lim_{y\to 0^+} u(0,y) &= -\frac{1}{2\pi}\lim_{y\to 0^+}\frac{36-y^2}{y}\ln\left(\frac{6+y}{6-y}\right)\\
&= -\frac{1}{2\pi}\lim_{y\to 0^+}\frac{\ln(\frac{6+y}{6-y})}{\frac{y}{36-y^2}}\\
&= -\frac{1}{2\pi}\lim_{y\to 0^+}\frac{\frac{d}{dy}(\ln(\frac{6+y}{6-y}))}{\frac{d}{dy}(\frac{y}{36-y^2})}\\
&= -\frac{1}{2\pi}\lim_{y\to 0^+}\frac{\frac{12}{36-y^2}}{\frac{36+y^2}{(36-y^2)^2}}\\
&= -\frac{6}{\pi}\lim_{y\to 0^+}\frac{36-y^2}{36+y^2}\\
&= -\frac{6}{\pi}\cdot 1\\
&= -\frac{6}{\pi}\\
&= u(0,0),
\end{aligned}$$

which shows that $u(0,y)$ is continuous for all $0 \le y < 6$.

(d) Using the separation of variables method, find the solution u in D.
Warning: This exercise is challenging!

Solution. Define $w(r,\theta) = u(x(r,\theta), y(r,\theta))$. Then the problem is transformed into

$$\begin{aligned}
&\Delta w = 0 && 0 < r < 6, 0 \le \theta \le 2\pi\\
&w(6,\theta) = \begin{cases} 6\cos(\theta) & \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2}\\ 0 & \text{if } 0 \le \theta \le \frac{\pi}{2}, \frac{3\pi}{2} \le \theta \le 2\pi. \end{cases}
\end{aligned}$$

For the Laplace equation on a disk, we have already done the separation of variables method in our solution to Exercise 7.7(b). As a result of the method, the general smooth solution of the Laplace equation on a disk is given by

$$w(r,\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta)).$$

Next, we will compute the Fourier coefficients $\alpha_0, \alpha_n, \beta_n$. We have

$$\begin{aligned}
w(6,\theta) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} 6^n(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta))\\
&= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} 6^n\alpha_n\cos(n\theta) + \sum_{n=1}^{\infty} 6^n\beta_n\sin(n\theta)
\end{aligned}$$

and the given boundary condition

$$w(6,\theta) = \begin{cases} 6\cos(\theta) & \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2}\\ 0 & \text{if } 0 \le \theta \le \frac{\pi}{2}, \frac{3\pi}{2} \le \theta \le 2\pi. \end{cases}$$

Because the coefficients are not constant for all $0 \le \theta \le 2\pi$, we are unable to equate the terms. Instead, we need to multiply by $1, \cos(\theta), \sin(\theta)$ in each case and integrate over $0 \le \theta \le 2\pi$ in order to compute the Fourier coefficients. That said, we have

$$\begin{aligned}
\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos(\theta)\,d\theta &= \int_0^{2\pi} w(6,\theta)\,d\theta\\
&= \int_0^{2\pi} \frac{\alpha_0}{2} + \sum_{m=1}^{\infty} 6^m\alpha_m\cos(m\theta) + \sum_{m=1}^{\infty} 6^m\beta_m\sin(m\theta)\,d\theta\\
&= \frac{\alpha_0}{2}\int_0^{2\pi} 1\,d\theta + \sum_{m=1}^{\infty} 6^m\alpha_m\int_0^{2\pi}\cos(m\theta)\,d\theta + \sum_{m=1}^{\infty} 6^m\beta_m\int_0^{2\pi}\sin(m\theta)\,d\theta\\
&= \frac{\alpha_0}{2}2\pi + \sum_{m=1}^{\infty} 6^m\alpha_m 0 + \sum_{m=1}^{\infty} 6^m\beta_m 0\\
&= \pi\alpha_0,
\end{aligned}$$

which implies

$$\begin{aligned}\alpha_0 &= \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos(\theta)\, d\theta \\ &= -\frac{12}{\pi}.\end{aligned}$$

We have

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos(\theta)\cos(n\theta)\, d\theta &= \int_0^{2\pi} w(6,\theta)\cos(n\theta)\, d\theta \\ &= \int_0^{2\pi}\left(\frac{\alpha_0}{2} + \sum_{m=1}^{\infty} 6^m\alpha_m\cos(m\theta) + \sum_{m=1}^{\infty} 6^m\beta_m\sin(m\theta)\right)\cos(n\theta)\, d\theta \\ &= \frac{\alpha_0}{2}\int_0^{2\pi}\cos(n\theta)\, d\theta + \sum_{m=1}^{\infty} 6^m\alpha_m\int_0^{2\pi}\cos(m\theta)\cos(n\theta)\, d\theta \\ &\quad + \sum_{m=1}^{\infty} 6^m\beta_m\int_0^{2\pi}\sin(m\theta)\cos(n\theta)\, d\theta \\ &= \frac{\alpha_0}{2}0 + \sum_{m=1}^{\infty} 6^m\alpha_m\begin{cases}\pi & \text{if } n = m,\\ 0 & \text{if } n \neq m\end{cases} + \sum_{m=1}^{\infty} 6^m\beta_m 0 \\ &= 6^n\alpha_n\pi,\end{aligned}$$

which implies

$$\begin{aligned}\alpha_n &= \frac{1}{\pi}6^{1-n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos(\theta)\cos(n\theta)\, d\theta \\ &= \frac{1}{\pi}6^{1-n}\begin{cases}\frac{\pi}{2} & \text{if } n = 1,\\ -\frac{2}{n^2-1} & \text{if } n = 2, 6, 10, \ldots,\\ 0 & \text{if } n = 3, 5, 7, \ldots,\\ \frac{2}{n^2-1} & \text{if } n = 4, 8, 12, \ldots\end{cases} \\ &= \begin{cases}\frac{1}{2} & \text{if } n = 1,\\ -\frac{2}{\pi}\frac{6^{1-n}}{n^2-1} & \text{if } n = 2, 6, 10, \ldots,\\ 0 & \text{if } n = 3, 5, 7, \ldots,\\ \frac{2}{\pi}\frac{6^{1-n}}{n^2-1} & \text{if } n = 4, 8, 12, \ldots.\end{cases}\end{aligned}$$

We have

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos(\theta)\sin(n\theta)\, d\theta &= \int_0^{2\pi} w(6,\theta)\sin(n\theta)\, d\theta \\ &= \int_0^{2\pi}\left(\frac{\alpha_0}{2} + \sum_{m=1}^{\infty} 6^m\alpha_m\cos(m\theta) + \sum_{m=1}^{\infty} 6^m\beta_m\sin(m\theta)\right)\sin(n\theta)\, d\theta \\ &= \frac{\alpha_0}{2}\int_0^{2\pi}\sin(n\theta)\, d\theta + \sum_{m=1}^{\infty} 6^m\alpha_m\int_0^{2\pi}\cos(m\theta)\sin(n\theta)\, d\theta \\ &\quad + \sum_{m=1}^{\infty} 6^m\beta_m\int_0^{2\pi}\sin(m\theta)\sin(n\theta)\, d\theta \\ &= \frac{\alpha_0}{2}0 + \sum_{m=1}^{\infty} 6^m\alpha_m 0 + \sum_{m=1}^{\infty} 6^m\beta_m\begin{cases}\pi & \text{if } n = m,\\ 0 & \text{if } n \neq m\end{cases} \\ &= 6^n\beta_n\pi,\end{aligned}$$

which implies

$$\begin{aligned}\beta_n &= \frac{1}{\pi}6^{1-n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos(\theta)\sin(n\theta)\, d\theta \\ &= \frac{1}{\pi}6^{1-n}0 \\ &= 0.\end{aligned}$$

Therefore, our formal solution in polar coordinates is

$$
\begin{aligned}
w(r,\theta) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n(\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)) \\
&= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} \beta_n r^n \sin(n\theta) \\
&= -\frac{\alpha_0}{\pi} + \alpha_1 r^1 \cos(1\theta) + \sum_{n=2,6,10,\ldots} \alpha_n r^n \cos(n\theta) + \sum_{n=3,5,7,\ldots} \alpha_n r^n \cos(n\theta) + \sum_{n=4,8,12,\ldots} \alpha_n r^n \cos(n\theta) \\
&\quad + \sum_{n=1}^{\infty} \beta_n r^n \sin(n\theta) \\
&= -\frac{6}{\pi} + \frac{1}{2} r^1 \cos(1\theta) - \frac{2}{\pi} \sum_{n=2,6,10,\ldots} \frac{6^{1-n}}{n^2-1} r^n \cos(n\theta) + \sum_{n=3,5,7,\ldots} 0 r^n \cos(n\theta) + \sum_{n=4,8,12,\ldots} \frac{2}{\pi} \frac{6^{1-n}}{n^2-1} r^n \cos(n\theta) \\
&\quad + \sum_{n=1}^{\infty} 0 r^n \sin(n\theta) \\
&= -\frac{6}{\pi} + \frac{1}{2} r \cos(\theta) - \frac{2}{\pi} \sum_{n=2,6,10,\ldots} \frac{6^{1-n}}{n^2-1} r^n \cos(n\theta) + \frac{2}{\pi} \sum_{n=4,8,12,\ldots} \frac{6^{1-n}}{n^2-1} r^n \cos(n\theta) \\
&= -\frac{6}{\pi} + \frac{1}{2} r \cos(\theta) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{6^{1-(4n-2)}}{(4n-2)^2-1} r^{4n-2} \cos((4n-2)\theta) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{6^{1-4n}}{(4n)^2-1} r^{4n} \cos(4n\theta),
\end{aligned}
$$

as desired. Finally, to convert this formula back into Cartesian coordinates, we will need to invoke the trigonometric identities

$$
\begin{aligned}
\cos(n\theta) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} \cos^{n-2k}(\theta) \sin^{2k}(\theta), \\
\sin(n\theta) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} \cos^{n-(2k+1)}(\theta) \sin^{2k+1}(\theta)
\end{aligned}
$$

for $n = 1, 2, 3, \ldots$, as seen on this question posted on Mathematics Stack Exchange. In Cartesian coordinates, our formal solution is

$$
\begin{aligned}
u(x,y) &= u(x(r,\theta), y(r,\theta)) \\
&= w(r,\theta) \\
&= -\frac{6}{\pi} + \frac{1}{2} r \cos(\theta) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{6^{1-(4n-2)}}{(4n-2)^2-1} r^{4n-2} \cos((4n-2)\theta) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{6^{1-4n}}{(4n)^2-1} r^{4n} \cos(4n\theta) \\
&= -\frac{6}{\pi} + \frac{1}{2} r \cos(\theta) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{6^{1-(4n-2)}}{(4n-2)^2-1} r^{4n-2} \sum_{k=0}^{\lfloor \frac{4n-2}{2} \rfloor} (-1)^k \binom{4n-2}{2k} \cos^{(4n-2)-2k}(\theta) \sin^{2k}(\theta) \\
&\quad + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{6^{1-4n}}{(4n)^2-1} r^{4n} \sum_{k=0}^{\lfloor \frac{4n}{2} \rfloor} (-1)^k \binom{4n}{2k} \cos^{4n-2k}(\theta) \sin^{2k}(\theta) \\
&= -\frac{6}{\pi} + \frac{1}{2} x - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{6^{1-(4n-2)}}{(4n-2)^2-1} (x^2+y^2)^{2n-1} \sum_{k=0}^{\lfloor \frac{4n-2}{2} \rfloor} (-1)^k \binom{4n-2}{2k} \left(\frac{x^2}{x^2+y^2}\right)^{2n-k-1} \left(\frac{y^2}{x^2+y^2}\right)^k \\
&\quad + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{6^{1-4n}}{(4n)^2-1} (x^2+y^2)^{2n} \sum_{k=0}^{\lfloor \frac{4n}{2} \rfloor} (-1)^k \binom{4n}{2k} \left(\frac{x^2}{x^2+y^2}\right)^{2n-k} \left(\frac{y^2}{x^2+y^2}\right)^k \\
&= \boxed{-\frac{6}{\pi} + \frac{1}{2} x - \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{k=0}^{2n-1} \frac{(-1)^k 6^{3-4n}}{16n^2-16n+3} \binom{4n-2}{2k} x^{4n-2k-2} y^{2k} + \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{k=0}^{2n} \frac{(-1)^k 6^{1-4n}}{16n^2-1} \binom{4n}{2k} x^{4n-2k} y^{2k}},
\end{aligned}
$$

where we used $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r^2 = x^2 + y^2$. ◻

Remark. *By substituting $x = 0$ into our expression of $u(x,y)$, we obtain*

$$
\begin{aligned}
u(0,y) &= -\frac{6}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} 6^{3-4n}}{16n^2-16n+3} \binom{4n-2}{2(n-1)} y^{2(n-1)} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{2n} 6^{1-4n}}{16n^2-1} \binom{4n}{2(2n)} y^{2(2n)} \\
&= -\frac{6}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{6^{3-4n}}{16n^2-16n+3} y^{4n-2} + \frac{6^{1-4n}}{16n^2-1} y^{4n} \right).
\end{aligned}
$$

It is possible to show that this expression is consistent with the expression

$$u(0,y) = \begin{cases} -\frac{6}{\pi} & \text{if } y = 0, \\ -\frac{1}{2\pi}\frac{36-y^2}{y}\ln\left(\frac{6+y}{6-y}\right) & \text{if } 0 < y < 6 \end{cases}$$

obtained from part (c). Indeed, the general Fourier series representation of $u(0,y)$ *over the integral* $0 \le y < 6$ *is*

$$u(0,y) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{6}y\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{6}y\right),$$

where A_0, A_n, B_n *are the Fourier coefficients given by*

$$\begin{aligned} A_0 &= \frac{2}{6}\int_0^6 u(0,y)\,dy = -\frac{1}{6\pi}\int_0^6 \frac{36-y^2}{y}\ln\left(\frac{6+y}{6-y}\right)dy, \\ A_n &= \frac{2}{6}\int_0^6 u(0,y)\cos\left(\frac{n\pi}{6}y\right)dy = -\frac{1}{6\pi}\int_0^6 \frac{36-y^2}{y}\ln\left(\frac{6+y}{6-y}\right)\cos\left(\frac{n\pi}{6}y\right)dy, \\ B_n &= \frac{2}{6}\int_0^6 u(0,y)\sin\left(\frac{n\pi}{6}y\right)dy = -\frac{1}{6\pi}\int_0^6 \frac{36-y^2}{y}\ln\left(\frac{6+y}{6-y}\right)\sin\left(\frac{n\pi}{6}y\right)dy. \end{aligned}$$

The procedure from here would be to compute explicitly A_0, A_n, B_n*, substitute these coefficients into the Fourier series representation of* $u(0,y)$*, and finally make some algebraic and trigonometric manipulations in order to arrive at the series expression of* $u(0,y)$ *that we wrote at the beginning of this remark. Nonetheless, this entire process is extremely tedious, and I have decided not to include it here in this remark or anywhere else in this homework solution. For what it is worth, you may view my saved graph on Desmos in order to verify by visual inspection that the graphs of our two final expressions of* $u(0,y)$ *over the interval* $-6 < y < 6$ *almost overlap each other. Note that, because Desmos is unable to compute the infinite series appearing in* $u(0,y)$*, I had to substitute it with a finite series with a large number of finitely many terms, such as* $N = 99$*, that approximates* $u(0,y)$*. If one were able to replace* $N = 99$ *with* $N = \infty$ *on Desmos, then the two graphs should be the same.*

(e) Is the solution classical?

Remark. *A solution is said to be* classical *if it is differentiable up to the highest-order term in the partial differential equation. In this case, the Laplace equation is a second-order partial differential equation. So we require that the solution* $u(x,y)$ *must be at least twice differentiable for all* $(x,y) \in D$ *in order it to be classical.*

Answer. I do not know the answer to this question, but I do know that one has to inspect our answer we obtained for part (d). A convergent sequence of smooth functions appearing as terms in a summation converges to a function that can be smooth, continuous, or even discontinuous. □