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Solutions to suggested homework problems from
An Introduction to Partial Differential Equations by Yehuda Pinchover and Jacob Rubinstein

Suggested problems: Exercises 8.5, 8.6, 8.7, 8.8, 8.11

8.5. (a) Show that the function

$$\begin{aligned} G(x,y;\xi,\eta) &:= \Gamma(x-\xi, y-\eta) - \Gamma(x-\tilde{\xi}, y-\tilde{\eta}) \\ &= -\frac{1}{2\pi}\ln\left(\frac{\sqrt{(x-\xi)^2+(y-\eta)^2}}{\sqrt{(x-\xi)^2+(y+\eta)^2}}\right) \end{aligned} \tag{8.23}$$

is indeed the Green function in $\mathbb{R}^2_+$, and that its derivative in the y direction for $y = 0$ is the Poisson kernel which is given by

$$K(x,0;\xi,\eta) := \frac{\eta}{\pi((x-\xi)^2+\eta^2)} \tag{8.24}$$

for all $(x,0) \in \partial\mathbb{R}^2_+$ and $(\xi,\eta) \in \partial\mathbb{R}^2_+$.

Solution. Recall that the fundamental solution of the Laplace equation with a pole at (ξ,η) is given by

$$\begin{aligned} \Gamma(x-\xi,y-\eta) &:= -\frac{1}{2\pi}\ln(\sqrt{(x-\xi)^2+(y-\eta)^2}) \\ &= -\frac{1}{4\pi}\ln((x-\xi)^2+(y-\eta)^2) \end{aligned}$$

and satisfies

$$\Delta\Gamma = -\delta(x-\xi,y-\eta)$$

for all $(x,y) \in \mathbb{R}^2_+$ with $(x,y) \neq (\xi,\eta)$, where

$$\delta(x-\xi,y-\eta) := \begin{cases} \infty & \text{if } (x,y) = (\xi,\eta), \\ 0 & \text{if } (x,y) \neq (\xi,\eta). \end{cases}$$

Given the upper half plane $\mathbb{R}^2_+$, the inverse point of (x,y) with respect to the real line is

$$(\tilde{x},\tilde{y}) := (x,-y).$$

By combining (8.13) with the boundary condition of (8.11) from the textbook, the possible expressions of the given function are

$$\begin{aligned} G(x,y;\xi,\eta) &:= \Gamma(x-\xi,y-\eta) - \Gamma(x-\tilde{\xi},y-\tilde{\eta}) \\ &= \Gamma(x-\xi,y-\eta) - \Gamma(x-\xi,y+\eta) \\ &= -\frac{1}{2\pi}\ln(\sqrt{(x-\xi)^2+(y-\eta)^2}) + \frac{1}{2\pi}\ln(\sqrt{(x-\xi)^2+(y+\eta)^2}) \\ &= -\frac{1}{2\pi}\ln\left(\frac{\sqrt{(x-\xi)^2+(y-\eta)^2}}{\sqrt{(x-\xi)^2+(y+\eta)^2}}\right). \end{aligned} \tag{8.23}$$

So we have

$$\begin{aligned} \Delta G(x,y;\xi,\eta) &= \Delta(\Gamma(x-\xi,y-\eta) - \Gamma(x-\xi,y+\eta)) \\ &= \Delta\Gamma(x-\xi,y-\eta) - \Delta\Gamma(x-\xi,y+\eta) \\ &= -\delta(x-\xi,y-\eta) - 0 \\ &= -\delta(x-\xi,y-\eta) \end{aligned}$$

and

$$\begin{aligned} G(x,0;\xi,\eta) &= -\frac{1}{2\pi}\ln\left(\frac{\sqrt{(x-\xi)^2+(0-\eta)^2}}{\sqrt{(x-\xi)^2+(0+\eta)^2}}\right) \\ &= -\frac{1}{2\pi}\ln\left(\frac{\sqrt{(x-\xi)^2+\eta^2}}{\sqrt{(x-\xi)^2+\eta^2}}\right) \\ &= -\frac{1}{2\pi}\ln(1) \\ &= -\frac{1}{2\pi}0 \\ &= 0. \end{aligned}$$

In summary, $G(x,y;\xi,\eta)$ solves

$$\begin{aligned} \Delta G &= -\delta(x-\xi, y-\eta) \quad && (x,y) \in \mathbb{R}^2_+, \\ G(x,0;\xi,\eta) &= 0 && (x,y) \in \partial\mathbb{R}^2_+, \end{aligned} \tag{8.14}$$

meaning that G is indeed the Green function. And the derivative of G in the y direction is

$$\begin{aligned} G_y(x,y;\xi,\eta) &= \frac{\partial}{\partial y}\left(-\frac{1}{2\pi}(\ln(\sqrt{(x-\xi)^2+(y-\eta)^2} - \ln(\sqrt{(x-\xi)^2+(y+\eta)^2})\right) \\ &= -\frac{1}{2\pi}\frac{\partial}{\partial y}(\ln(\sqrt{(x-\xi)^2+(y-\eta)^2}) + \frac{1}{2\pi}\frac{\partial}{\partial y}(\ln(\sqrt{(x-\xi)^2+(y+\eta)^2}) \\ &= -\frac{1}{2\pi}\frac{y-\eta}{(x-\xi)^2+(y-\eta)^2} + \frac{1}{2\pi}\frac{y+\eta}{(x-\xi)^2+(y+\eta)^2}. \end{aligned}$$

At $y=0$, we obtain

$$\begin{aligned} G_y(x,0;\xi,\eta) &= -\frac{1}{2\pi}\frac{0-\eta}{(x-\xi)^2+(0-\eta)^2} + \frac{1}{2\pi}\frac{0+\eta}{(x-\xi)^2+(0+\eta)^2} \\ &= \frac{1}{2}\frac{\eta}{\pi((x-\xi)^2+\eta^2)} + \frac{1}{2}\frac{\eta}{\pi((x-\xi)^2+\eta^2)} \\ &= \frac{\eta}{\pi((x-\xi)^2+\eta^2)} \\ &= K(x,0;\xi,\eta) \end{aligned}$$

for all $(x,0) \in \partial\mathbb{R}^2_+$ and $(\xi,\eta) \in \partial\mathbb{R}^2_+$, as desired. □

(b) Using a reflection principle and part (a), find the Green function of the positive quarter plane $\{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$.

Solution. Given the positive quarter plane $D := \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, the inverse points of (x,y) with respect to ∂D are

$$\begin{aligned} (x_1,y_1) &:= (x,-y), \\ (x_2,y_2) &:= (-x,y), \\ (x_3,y_3) &:= (-x,-y). \end{aligned}$$

Using the reflection principle, we have the function

$$\begin{aligned} G(x,y;\xi,\eta) &= \Gamma(x-\xi,y-\eta) - \Gamma(x-\xi_1,y-\eta_1) - \Gamma(x-\xi_2,y-\eta_2) + \Gamma(x-\xi_3,y-\eta_3) \\ &= \Gamma(x-\xi,y-\eta) - \Gamma(x-\xi,y+\eta) - \Gamma(x+\xi,y-\eta) + \Gamma(x+\xi,y+\eta) \\ &= -\frac{1}{2\pi}\ln(\sqrt{(x-\xi)^2+(y-\eta)^2}) + \frac{1}{2\pi}\ln(\sqrt{(x-\xi)^2+(y+\eta)^2}) \\ &\quad + \frac{1}{2\pi}\ln(\sqrt{(x+\xi)^2+(y-\eta)^2}) - \frac{1}{2\pi}\ln(\sqrt{(x+\xi)^2+(y+\eta)^2}) \\ &= -\frac{1}{4\pi}\ln((x-\xi)^2+(y-\eta)^2) + \frac{1}{4\pi}\ln((x-\xi)^2+(y+\eta)^2) \\ &\quad + \frac{1}{4\pi}\ln((x+\xi)^2+(y-\eta)^2) - \frac{1}{4\pi}\ln((x+\xi)^2+(y+\eta)^2) \\ &= \boxed{-\frac{1}{4\pi}\ln\left(\frac{((x-\xi)^2+(y-\eta)^2)((x+\xi)^2+(y+\eta)^2)}{((x-\xi)^2+(y+\eta)^2)((x+\xi)^2+(y-\eta)^2)}\right)}. \end{aligned}$$

Now, it remains to show that this function is indeed the Green function. We have

$$\begin{aligned} \Delta G(x,y;\xi,\eta) &= \Delta\Gamma(x-\xi,y-\eta) - \Delta\Gamma(x-\xi_1,y-\eta_1) - \Delta\Gamma(x-\xi_2,y-\eta_2) + \Delta\Gamma(x-\xi_3,y-\eta_3) \\ &= -\delta(x-\xi,y-\eta) - 0 - 0 + 0 \\ &= -\delta(x-\xi,y-\eta). \end{aligned}$$

We also have, for all $x \geq 0$,

$$\begin{aligned} G(x,0;\xi,\eta) &= -\frac{1}{4\pi}\ln\left(\frac{((x-\xi)^2+(0-\eta)^2)((x+\xi)^2+(0+\eta)^2)}{((x-\xi)^2+(0+\eta)^2)((x+\xi)^2+(0-\eta)^2)}\right) \\ &= -\frac{1}{4\pi}\ln\left(\frac{((x-\xi)^2+\eta^2)((x+\xi)^2+\eta^2)}{((x-\xi)^2+\eta^2)((x+\xi)^2+\eta^2)}\right) \\ &= -\frac{1}{4\pi}\ln(1) \\ &= -\frac{1}{4\pi}0 \\ &= 0 \end{aligned}$$

and, for all $y \geq 0$,

$$
\begin{aligned}
G(0, y; \xi, \eta) &= -\frac{1}{4\pi} \ln\left(\frac{((0-\xi)^2 + (y-\eta)^2)((0+\xi)^2 + (y+\eta)^2)}{((0-\xi)^2 + (y+\eta)^2)((0+\xi)^2 + (y-\eta)^2)}\right) \\
&= -\frac{1}{4\pi} \ln\left(\frac{(\xi^2 + (y-\eta)^2)(\xi^2 + (y+\eta)^2)}{(\xi^2 + (y+\eta)^2)(\xi^2 + (y-\eta)^2)}\right) \\
&= -\frac{1}{4\pi} \ln(1) \\
&= -\frac{1}{4\pi} 0 \\
&= 0.
\end{aligned}
$$

In summary, $G(x, y; \xi, \eta)$ solves

$$
\begin{aligned}
\Delta G(x, y; \xi, \eta) &= -\delta(x-\xi, y-\eta) \quad && (x, y) \in D, \\
G(x, y; \xi, \eta) &= 0 && (x, y) \in \partial D,
\end{aligned}
\tag{8.14}
$$

meaning that G is indeed the Green function. □

8.6. Let $\mathbb{R}^2_+$ be the upper half-plane. Find the Neumann function of $\mathbb{R}^2_+$.

Solution. Given the upper half-plane $\mathbb{R}^2_+$, the inverse point of (x, y) with respect to the real line is

$$(\tilde{x}, \tilde{y}) := (x, -y).$$

The possible expressions of the given function are

$$
\begin{aligned}
N(x, y; \xi, \eta) &:= \Gamma(x-\xi, y-\eta) + \Gamma(x-\tilde{\xi}, y-\tilde{\eta}) + C \\
&= \Gamma(x-\xi, y-\eta) + \Gamma(x-\xi, y+\eta) + C \\
&= -\frac{1}{2\pi} \ln(\sqrt{(x-\xi)^2 + (y-\eta)^2}) - \frac{1}{2\pi} \ln(\sqrt{(x-\xi)^2 + (y+\eta)^2}) + C \\
&= -\frac{1}{2\pi} \ln(\sqrt{(x-\xi)^2 + (y-\eta)^2}\sqrt{(x-\xi)^2 + (y+\eta)^2}) + C,
\end{aligned}
$$

where C is a constant. This expression is based on the fact that one must place another positive charge (instead of negative charge for Dirichlet case) on the image point $(\tilde{\xi}, \tilde{\eta}) = (\xi, -\eta)$, in order to satisfy the Neumann condition $\partial_n N(x, y; \xi, \eta) = 0 = -\frac{1}{L}$ with $L = \infty$. We have

$$
\begin{aligned}
\Delta N(x, y; \xi, \eta) &= \Delta(\Gamma(x-\xi, y-\eta) - \Gamma(x-\xi, y+\eta) + C) \\
&= \Delta\Gamma(x-\xi, y-\eta) - \Delta\Gamma(x-\xi, y+\eta) - \Delta C \\
&= -\delta(x-\xi, y-\eta) - 0 + 0 \\
&= -\delta(x-\xi, y-\eta)
\end{aligned}
$$

and, for all $(x, 0; \xi, \eta) \in \partial\mathbb{R}^2_+$,

$$
\begin{aligned}
\partial_n(x, 0; \xi, \eta) &= \partial_n\left(-\frac{1}{2\pi} \ln(\sqrt{(x-\xi)^2 + (0-\eta)^2}\sqrt{(x-\xi)^2 + (0+\eta)^2}) + C\right) \\
&= -\frac{1}{2\pi}\partial_n \ln(\sqrt{(x-\xi)^2 + \eta^2}\sqrt{(x-\xi)^2 + \eta^2}) - \partial_n C \\
&= -\frac{1}{2\pi}\partial_n \ln((x-\xi)^2 + \eta^2) \\
&= -\frac{1}{2\pi}\nabla(\ln((x-\xi)^2 + \eta^2)) \cdot \hat{n} \\
&= -\frac{1}{2\pi}\left(\frac{\partial}{\partial x}(\ln((x-\xi)^2 + \eta^2)), \frac{\partial}{\partial y}(\ln((x-\xi)^2 + \eta^2))\right) \cdot (0, -1) \\
&= \frac{1}{2\pi}\frac{\partial}{\partial y} \ln((x-\xi)^2 + \eta^2) \\
&= \frac{1}{2\pi} 0 \\
&= 0 \\
&= -\frac{1}{L}
\end{aligned}
$$

with $L = \infty$. In summary, $N(x, y; \xi, \eta)$ solves

$$
\begin{aligned}
\Delta N(x, y; \xi, \eta) &= -\delta(x - \xi, y - \eta) \quad && (x, y) \in \mathbb{R}^2_+, \\
\partial_n N(x, 0; \xi, \eta) &= -\frac{1}{L} && (x, y) \in \partial\mathbb{R}^2_+,
\end{aligned}
\tag{8.29}
$$

meaning that G is indeed the Neumann function. □

8.7. (a) Let u be a smooth function with a compact support in $\mathbb{R}^2$. Prove

$$
\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) u(\vec{x})\, d\vec{x} = u(\vec{y}) = \int_{\mathbb{R}^2} \delta(\vec{x} - \vec{y}) u(\vec{x})\, d\vec{x}. \tag{8.9}
$$

Proof. Recall from page 212 of the textbook that $\rho_\epsilon(\vec{x})$ has compact support in $B(0, \epsilon)$ and satisfies

$$
\int_{\mathbb{R}^2} \rho_\epsilon(\vec{x})\, d\vec{x} = \int_{B(0,\epsilon)} \rho_\epsilon(\vec{x})\, d\vec{x} = 1,
$$

which implies

$$
\begin{aligned}
\int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) u(\vec{x})\, d\vec{x} - u(\vec{y}) &= \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) u(\vec{x})\, d\vec{x} - u(\vec{y}) \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x})\, d\vec{x} \\
&= \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) u(\vec{x})\, d\vec{x} - \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) u(\vec{y})\, d\vec{x} \\
&= \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) u(\vec{x}) - \rho_\epsilon(\vec{x}) u(\vec{y})\, d\vec{x} \\
&= \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) (u(\vec{x}) - u(\vec{y}))\, d\vec{x}.
\end{aligned}
$$

Since u is smooth in $\mathbb{R}^2$, it is continuous at $y \in \mathbb{R}^2$. So we can invoke the ϵ-δ definition of continuity, which states: For all $\epsilon > 0$, there exists $\delta > 0$ that satisfies $|u(x) - u(y)| < \epsilon$ for all $y \in B_\epsilon(x)$. Applying this definition and using the triangle inequality for integrals, we have

$$
\begin{aligned}
\left| \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) u(\vec{x})\, d\vec{x} - u(\vec{y}) \right| &= \left| \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) (u(\vec{x}) - u(\vec{y}))\, d\vec{x} \right| \\
&\le \int_{\mathbb{R}^2} |\rho_\epsilon(\vec{x}) u(\vec{x}) - u(\vec{y})|\, d\vec{x} \\
&= \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) |u(\vec{x}) - u(\vec{y})|\, d\vec{x} \\
&< \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) \epsilon\, d\vec{x} \\
&< \epsilon \int_{\mathbb{R}^2} \rho_\epsilon(\vec{x})\, d\vec{x} \\
&= \epsilon \cdot 1 \\
&= \epsilon \\
&\to 0
\end{aligned}
$$

as $\epsilon \to 0^+$. This implies

$$
\int_{\mathbb{R}^2} \rho_\epsilon(\vec{x}) u(\vec{x})\, d\vec{x} \to u(y)
$$

as $\epsilon \to 0^+$, as desired. □

(b) Find the constant c in

$$
\rho(\vec{x}) := \begin{cases} c \exp(\frac{1}{|\vec{x}|^2 - 1}) & \text{if } |\vec{x}| \le 1, \\ 0 & \text{if } |\vec{x}| > 1, \end{cases} \tag{8.10}
$$

and verify directly that ρ_ϵ is an approximation of the Dirac delta function.

Solution. We need to find the constant c that satisfies

$$
\int_{\mathbb{R}^2} \rho(\vec{x})\, d\vec{x} = 1.
$$

We have

$$\begin{aligned} 1 = \int_{\mathbb{R}^2} \rho(\vec{x})\, d\vec{x} = c \int_{B(0,1)} \exp\left(\frac{1}{|\vec{x}|^2 - 1}\right) d\vec{x} \\ = c \int_0^{2\pi} \int_0^1 \exp\left(\frac{1}{r^2-1}\right) r\, dr\, d\theta \\ = 2\pi c \int_0^1 \exp\left(\frac{1}{r^2-1}\right) r\, dr \\ \approx 0.466512c, \end{aligned}$$

where the approximate value is taken from this computation on WolframAlpha. We find $c \approx \frac{1}{0.466512} = \boxed{2.143566}$. ◻

Remark. *Although the integral*

$$\int_0^1 \exp\left(\frac{1}{r^2-1}\right) r\, dr$$

is convergent, we are unable to evaluate it analytically. We can try to invoke the Taylor series of the exponential function to obtain

$$\int_0^1 \exp\left(\frac{1}{r^2-1}\right) r\, dr = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \frac{r}{(r^2-1)^n}\, dr,$$

but the Taylor series entails an antiderivative that contains a divergent term. Our only recourse is to resort to numerical methods or advanced calculators in order to approximate the integral, which means we can only approximate the value of c at best.

8.8. Let $k \neq 0$. Show that the function $G_k(x;\xi) = \frac{1}{2k} e^{-k|x-\xi|}$ is a fundamental solution of the equation

$$-u'' + k^2 u = 0$$

for all $-\infty < x < \infty$.

Hint: Use one of Green's identities.

Proof. Given the function

$$\begin{aligned} G_k(x;\xi) := \frac{1}{2k} e^{-k|x-\xi|} \\ = \begin{cases} \frac{1}{2k} e^{-k(x-\xi)} & \text{if } x \geq \xi, \\ \frac{1}{2k} e^{k(x-\xi)} & \text{if } x < \xi, \end{cases} \end{aligned}$$

we obtain its first and second derivatives

$$G_k'(x;\xi) = \begin{cases} -\frac{1}{2} e^{-k(x-\xi)} & \text{if } x > \xi, \\ \frac{1}{2} e^{k(x-\xi)} & \text{if } x < \xi, \end{cases}$$

$$G_k''(x;\xi) = \begin{cases} \frac{k}{2} e^{-k(x-\xi)} & \text{if } x > \xi, \\ \frac{k}{2} e^{k(x-\xi)} & \text{if } x < \xi. \end{cases}$$

For all $x \neq \xi$, we have

$$\begin{aligned} -G_k''(x;\xi) + k^2 G_k(x;\xi) = -\begin{cases} \frac{k}{2} e^{-k(x-\xi)} & \text{if } x > \xi, \\ \frac{k}{2} e^{k(x-\xi)} & \text{if } x < \xi. \end{cases} + k^2 \begin{cases} \frac{1}{2k} e^{-k(x-\xi)} & \text{if } x > \xi, \\ \frac{1}{2k} e^{k(x-\xi)} & \text{if } x < \xi, \end{cases} \\ = \begin{cases} -\frac{k}{2} e^{-k(x-\xi)} + \frac{k}{2} e^{-k(x-\xi)} & \text{if } x > \xi, \\ -\frac{k}{2} e^{k(x-\xi)} + \frac{k}{2} e^{k(x-\xi)} & \text{if } x < \xi, \end{cases} \\ = \begin{cases} 0 & \text{if } x > \xi, \\ 0 & \text{if } x < \xi, \end{cases} \\ = 0, \end{aligned}$$

which implies in particular $-G_k''(x;\xi) + k^2 G_k(x;\xi) = 0$ for all $x \in (-\infty, x-\epsilon) \cup (x+\epsilon, \infty)$, where $\epsilon > 0$ is arbitrary. By using Green's third identity (integration by parts), we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} u(x)(-G_k''(x;\xi)+k^2G_k(x;\xi))\,dx &= \int_{\xi-\epsilon}^{\xi+\epsilon} u(x)(-G_k''(x;\xi)+k^2G_k(x;\xi))\,dx \\
&\quad + \int_{(-\infty,x-\epsilon)\cup(x+\epsilon,\infty)} u(x)(-G_k''(x;\xi)+k^2G_k(x;\xi))\,dx \\
&= \int_{\xi-\epsilon}^{\xi+\epsilon} u(x)(-G_k''(x;\xi)+k^2G_k(x;\xi))\,dx + \int_{(-\infty,x-\epsilon)\cup(x+\epsilon,\infty)} u(x)\cdot 0\,dx \\
&= \int_{\xi-\epsilon}^{\xi+\epsilon} u(x)(-G_k''(x;\xi)+k^2G_k(x;\xi))\,dx \\
&= -\int_{\xi-\epsilon}^{\xi+\epsilon} u(x)G_k''(x;\xi)\,dx + \int_{\xi-\epsilon}^{\xi+\epsilon} k^2u(x)G_k(x;\xi)\,dx \\
&= -\left(u(x)G_k''(x;\xi)\Big|_{\xi-\epsilon}^{\xi+\epsilon} + \int_{\xi-\epsilon}^{\xi+\epsilon} u'(x)G_k'(x;\xi)\,dx\right) \\
&\quad + \int_{\xi-\epsilon}^{\xi+\epsilon} k^2u(x)G_k(x;\xi)\,dx \\
&= -(u(\xi+\epsilon)G_k'(\xi+\epsilon;\xi) - u(\xi-\epsilon)G_k'(\xi-\epsilon;\xi)) \\
&\quad + \int_{\xi-\epsilon}^{\xi+\epsilon} -u'(x)G_k'(x;\xi)+k^2u(x)G_k(x;\xi)\,dx \\
&=: A_\epsilon + B_\epsilon.
\end{aligned}
$$

We have

$$
\begin{aligned}
A_\epsilon &= -(u(\xi+\epsilon)G_k'(\xi+\epsilon;\xi) - u(\xi-\epsilon)G_k'(\xi-\epsilon;\xi)) \\
&= -u(\xi+\epsilon)G_k'(\xi+\epsilon;\xi) + u(\xi-\epsilon)G_k'(\xi-\epsilon;\xi) \\
&= -u(\xi+\epsilon)\left(-\frac{1}{2}e^{-k((\xi+\epsilon)-\xi)}\right) + u(\xi-\epsilon)\left(\frac{1}{2}e^{k((\xi-\epsilon)-\xi)}\right) \\
&= \frac{1}{2}u(\xi+\epsilon)e^{-k\epsilon} + \frac{1}{2}u(\xi-\epsilon)e^{-k\epsilon} \\
&\to \frac{1}{2}u(\xi) + \frac{1}{2}u(\xi) \\
&= u(\xi)
\end{aligned}
$$

as $\epsilon \to 0^+$. As $u(x)$ and $G_k(x;\xi)$ and their first derivatives are bounded, we have

$$
\begin{aligned}
|u(x)| &\le C_1, \\
|u'(x)| &\le C_2, \\
|G_k(x;\xi)| &\le D_1, \\
|G_k'(x;\xi)| &\le D_2,
\end{aligned}
$$

where C_1, C_2, D_1, D_2 are constants, and so, by the triangle inequality and the triangle inequality for integrals, we have

$$
\begin{aligned}
|B_\epsilon| &= \left|\int_{\xi-\epsilon}^{\xi+\epsilon} -u'(x)G_k'(x;\xi)+k^2u(x)G_k(x;\xi)\,dx\right| \\
&\le \int_{\xi-\epsilon}^{\xi+\epsilon} |-u'(x)G_k'(x;\xi)+k^2u(x)G_k(x;\xi)|\,dx \\
&\le \int_{\xi-\epsilon}^{\xi+\epsilon} |-u'(x)G_k'(x;\xi)| + |k^2u(x)G_k(x;\xi)|\,dx \\
&= \int_{\xi-\epsilon}^{\xi+\epsilon} |u'(x)||G_k'(x;\xi)| + k^2|u(x)G_k(x;\xi)|\,dx \\
&\le \int_{\xi-\epsilon}^{\xi+\epsilon} C_2D_2 + k^2C_1D_1\,dx \\
&= 2(C_2D_2+k^2C_1D_1)\epsilon \\
&\to 0,
\end{aligned}
$$

which implies $B_\epsilon \to 0$, as $\epsilon \to 0^+$. Therefore, we conclude

$$\begin{aligned}\int_{-\infty}^{\infty} u(x)(-G_k''(x;\xi) + k^2 G_k(x;\xi))\,dx &= \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} u(x)(-G_k''(x;\xi) + k^2 G_k(x;\xi))\,dx \\ &= \lim_{\epsilon\to 0^+} (A_\epsilon + B_\epsilon) \\ &= \lim_{\epsilon\to 0^+} A_\epsilon + \lim_{\epsilon\to 0^+} B_\epsilon \\ &= u(\xi) + 0 \\ &= u(\xi),\end{aligned}$$

and so, according to Definition 8.3(b) in pages 212-213 of the textbook, $G_k(x;\xi)$ satisfies

$$-G_k''(x;\xi) + k^2 G_k(x;\xi) = \delta(x-\xi)$$

for all $-\infty < x < \infty$, which means that $G_k(x;\xi)$ is a fundamental solution of the equation $-u'' + k^2 u = 0$. □

8.11. Let $D_R := \mathbb{R}^2 \setminus B_R$ be the exterior of the disk with radius R centered at the origin. Find the (Dirichlet) Green function of D_R.

Remark. *For my solution below, I am following Example 8.14 of the textbook with necessary modifications for D_R.*

Solution. Let $(x,y) \in D_R$. The point

$$(\tilde{x}, \tilde{y}) := \frac{R^2}{x^2+y^2}(x,y) = \left(\frac{R^2}{x^2+y^2}x, \frac{R^2}{x^2+y^2}y\right)$$

is the inverse point of (x,y) with respect to the circle ∂D_R. Define the function

$$G_R(x,y;\xi,\eta) := \Gamma(x-\xi, y-\eta) - \Gamma\left(\frac{\sqrt{\xi^2+\eta^2}}{R}(x-\tilde{\xi}, y-\tilde{\eta})\right)$$

and set

$$\begin{aligned} r &= \sqrt{(x-\xi)^2 + (y-\eta)^2}, \\ r^* &= \sqrt{\left(x - \frac{R^2}{\rho^2}\xi\right)^2 + \left(y - \frac{R^2}{\rho^2}\eta\right)^2}, \\ \rho &= \sqrt{\xi^2+\eta^2}. \end{aligned}$$

Then we have

$$\begin{aligned} G_R(x,y;\xi,\eta) &= \Gamma(x-\xi, y-\eta) - \Gamma\left(\frac{\sqrt{\xi^2+\eta^2}}{R}(x-\tilde{\xi}, y-\tilde{\eta})\right) \\ &= \Gamma(x-\xi, y-\eta) - \Gamma\left(\frac{\sqrt{\xi^2+\eta^2}}{R}\left(x - \frac{R^2}{\xi^2+\eta^2}\xi\right), \frac{\sqrt{\xi^2+\eta^2}}{R}\left(y - \frac{R^2}{\xi^2+\eta^2}\eta\right)\right) \\ &= \Gamma(x-\xi, y-\eta) - \Gamma\left(\frac{\rho}{R}\left(x - \frac{R^2}{\rho^2}\xi\right), \frac{\rho}{R}\left(y - \frac{R^2}{\rho^2}\eta\right)\right) \\ &= -\frac{1}{2\pi}\ln(\sqrt{(x-\xi)^2+(y-\eta)^2}) + \frac{1}{2\pi}\ln\left(\sqrt{\frac{\rho^2}{R^2}\left(x-\frac{R^2}{\rho^2}\xi\right)^2 + \frac{\rho^2}{R^2}\left(y-\frac{R^2}{\rho^2}\eta\right)^2}\right) \\ &= -\frac{1}{2\pi}\ln(\sqrt{(x-\xi)^2+(y-\eta)^2}) + \frac{1}{2\pi}\ln\left(\frac{\rho}{R}\sqrt{\left(x-\frac{R^2}{\rho^2}\xi\right)^2 + \left(y-\frac{R^2}{\rho^2}\eta\right)^2}\right) \\ &= -\frac{1}{2\pi}\ln(r) + \frac{1}{2\pi}\ln\left(\frac{\rho r^*}{R}\right) \\ &= -\frac{1}{2\pi}\left(\ln(r) - \ln\left(\frac{\rho r^*}{R}\right)\right) \\ &= -\frac{1}{2\pi}\ln\left(\frac{Rr}{\rho r^*}\right) \end{aligned}$$

for all $(x,y) \neq (\xi,\eta)$. Now, it remains to show that this function is indeed the Green function. We have

$$\begin{aligned} \Delta G_R(x,y;\xi,\eta) &= \Delta\left(\Gamma(x-\xi,y-\eta) - \Gamma\left(\frac{\rho}{R}\left(x-\frac{R^2}{\rho^2}\xi\right), \frac{\rho}{R}\left(y-\frac{R^2}{\rho^2}\eta\right)\right)\right) \\ &= \Delta\Gamma(x-\xi,y-\eta) - \Delta\Gamma\left(\frac{\rho}{R}\left(x-\frac{R^2}{\rho^2}\xi\right), \frac{\rho}{R}\left(y-\frac{R^2}{\rho^2}\eta\right)\right) \\ &= -\delta(x-\xi, y-\delta) - 0 \\ &= -\delta(x-\xi, y-\delta). \end{aligned}$$

For all $(x, y) \in \partial D_R$ (that is, $x^2 + y^2 = R^2$), we have

$$\begin{aligned}
(\tilde{x}, \tilde{y}) &:= \left(\frac{R^2}{x^2 + y^2} x, \frac{R^2}{x^2 + y^2} y \right) \\
&= \left(\frac{R^2}{R^2} x, \frac{R^2}{R^2} y \right) \\
&= (x, y),
\end{aligned}$$

and so we have

$$\begin{aligned}
G_R(x, y; \xi, \eta) &= \Gamma(x - \xi, y - \eta) - \Gamma\left(\frac{\sqrt{\xi^2 + \eta^2}}{R} (x - \tilde{\xi}, y - \tilde{\eta}) \right) \\
&= \Gamma(x - \xi, y - \eta) - \Gamma\left(\frac{R}{R} (x - \xi, y - \eta) \right) \\
&= \Gamma(x - \xi, y - \eta) - \Gamma(x - \xi, y - \eta) \\
&= 0.
\end{aligned}$$

In summary, $G(x, y; \xi, \eta)$ solves

$$\begin{aligned}
\Delta G &= -\delta(x - \xi, y - \eta) \quad && (x, y) \in D, \\
G(x, y; \xi, \eta) &= 0 && (x, y) \in \partial D,
\end{aligned} \tag{8.14}$$

meaning that G is indeed the Green function. □