MATH 146C discussion
University of California, Riverside

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Solutions to suggested homework problems from
An Introduction to Partial Differential Equations by Yehuda Pinchover and Jacob Rubinstein

Suggested problems: Exercises 9.3, 9.4, 9.5, 9.6, 9.7

9.3. Derive the radially symmetric solution

$$u(r,t) = \frac{1}{2r}((r+ct)\tilde{f}(r+ct) + (r-ct)\tilde{f}(r-ct)) + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\tilde{g}(s)\,ds \tag{9.26}$$

for the three-dimensional (radial) wave equation

$$u_{tt} - c^2\Delta u = 0,$$

where $\tilde{f}$ and $\tilde{g}$, defined by

$$\tilde{f}(r) := \begin{cases} f(r) & \text{if } r \geq 0, \\ f(-r) & \text{if } r < 0, \end{cases}$$

$$\tilde{g}(r) := \begin{cases} g(r) & \text{if } r \geq 0, \\ g(-r) & \text{if } r < 0, \end{cases}$$

are the even extensions of f and g, respectively.

Remark. *Portions of my entire solution below for this exercise is also outlined in your professor's Lecture 17 (May 29) notes or in Sections 4.2, 4.3, and 9.4 of the textbook.*

Solution - Step 1: Derive the formal solution of the one-dimensional wave equation with initial conditions. We would like to solve the Cauchy problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \qquad -\infty < x < \infty, t > 0, \\ u(x,0) &= f(x) \quad -\infty < x < \infty, \\ u_t(x,0) &= g(x) \quad -\infty < x < \infty. \end{aligned}$$

Define the new variables $\xi := x + ct$ and $\eta := x - ct$, which implies the first partial derivatives

$$\begin{aligned} \xi_x &= (x+ct)_x = 1, \\ \xi_t &= (x+ct)_t = c, \\ \eta_x &= (x-ct)_x = 1, \\ \eta_t &= (x-ct)_t = -c. \end{aligned}$$

Also set $w(\xi,\eta) := u(x(\xi,\eta), y(\xi,\eta))$. Then, by the multivariable chain rule, we obtain the first partial derivatives

$$\begin{aligned} u_x &= w_\xi \xi_x + w_\eta \eta_x \\ &= w_\xi \cdot 1 + w_\eta \cdot 1 \\ &= w_\xi + w_\eta \end{aligned}$$

and

$$\begin{aligned} u_t &= w_\xi \xi_t + w_\eta \eta_t \\ &= w_\xi c + w_\eta(-c) \\ &= c(w_\xi - w_\eta), \end{aligned}$$

as well as the second partial derivatives

$$\begin{aligned} u_{xx} &= (w_\xi + w_\eta)_x \\ &= (w_\xi + w_\eta)_\xi \xi_x + (w_\xi + w_\eta)_\eta \eta_x \\ &= (w_{\xi\xi} + w_{\xi\eta}) \cdot 1 + (w_{\xi\eta} + w_{\eta\eta}) \cdot 1 \\ &= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta} \end{aligned}$$

and

$$\begin{aligned} u_{tt} &= c(w_\xi - w_\eta)_t \\ &= c(w_\xi - w_\eta)_\xi \xi_t + c(w_\xi - w_\eta)_\eta \eta_t \\ &= c(w_{\xi\xi} - w_{\xi\eta})c + c(w_{\xi\eta} - w_{\eta\eta})(-c) \\ &= c^2(w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta}). \end{aligned}$$

So we have

$$\begin{aligned}
0 &= u_{tt} - c^2 u_{xx} \\
&= c^2(w_{\xi\xi} - 2w_{\xi\eta} - w_{\eta\eta}) - c^2(w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}) \\
&= c^2(\cancel{w_{\xi\xi}} - 2w_{\xi\eta} - \cancel{w_{\eta\eta}} - \cancel{w_{\xi\xi}} - 2w_{\xi\eta} + \cancel{w_{\eta\eta}}) \\
&= -4c^2 w_{\xi\eta},
\end{aligned}$$

which implies

$$w_{\xi\eta} = 0.$$

This partial differential equation has the general solution

$$\begin{aligned}
w(\xi, \eta) &= \int \left(\int w_{\xi\eta} \, d\eta \right) d\xi \\
&= \int \left(\int 0 \, d\eta \right) d\xi \\
&= \int H(\xi) d\xi \\
&= F(\xi) + G(\eta).
\end{aligned}$$

where $F(\xi), H(\xi)$ are arbitrary functions of ξ and $G(\eta)$ is an arbitrary function of η. Therefore, we have

$$\begin{aligned}
u(x, t) &= u(x(\xi, \eta), t(\xi, \eta)) \\
&= w(\xi, \eta) \\
&= F(\xi) + G(\eta) \\
&= F(x + ct) + G(x - ct),
\end{aligned}$$

as well as one of its partial derivatives

$$\begin{aligned}
u_t(x, t) &= (F(x + ct) + G(x - ct))_t \\
&= (F(x + ct))_t + (G(x - ct))_t \\
&= F'(x + ct)(x + ct)_t + G'(x - ct)(x - ct)_t \\
&= F'(x + ct)c + G'(x - ct)(-c) \\
&= c(F'(x + ct) - G'(x - ct)).
\end{aligned}$$

At $t = 0$, we have

$$\begin{aligned}
f(x) &= u(x, 0) = F(x) + G(x), \\
g(x) &= u_t(x, 0) = c(F'(x) - G'(x)),
\end{aligned}$$

or equivalently the linear system of equations

$$\begin{aligned}
F(x) + G(x) &= f(x), \\
F(x) - G(x) &= \frac{1}{c} \int_0^x g(s) \, ds + C,
\end{aligned}$$

where C is a constant. We can solve simultaneously the system of equations to obtain

$$\begin{aligned}
F(x) &= \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(s) \, ds + \frac{C}{2}, \\
G(x) &= \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(s) \, ds - \frac{C}{2}.
\end{aligned}$$

Therefore, our formal solution is

$$\begin{aligned}
u(x, t) &= F(x + ct) + G(x - ct) \\
&= \left(\frac{1}{2} f(x + ct) + \frac{1}{2c} \int_0^{x+ct} g(s) \, ds + \cancel{\frac{C}{2}} \right) + \left(\frac{1}{2} f(x - ct) - \frac{1}{2c} \int_0^{x-ct} g(s) \, ds - \cancel{\frac{C}{2}} \right) \\
&= \frac{1}{2} f(x + ct) + \frac{1}{2} f(x - ct) + \frac{1}{2c} \int_0^{x+ct} g(s) \, ds + \frac{1}{2c} \int_{x-ct}^0 g(s) \, ds \\
&= \frac{1}{2}(f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds,
\end{aligned}$$

which is also known as *d'Alembert's formula*. □

Solution - Step 2: Derive the radially symmetric solution for the three-dimensional wave equation. Given any arbitrary point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, set $r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} \geq 0$. The Laplacian on $\mathbb{R}^3$ in spherical coordinates is

$$\begin{aligned}
\Delta u &:= u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} \\
&= u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(\frac{\cos(\phi)}{\sin(\phi)}u_\phi + u_{\phi\phi} + \frac{1}{\sin^2(\phi)}u_{\theta\theta}\right) \\
&= \frac{1}{r^2}(r^2u_r)_r + \frac{1}{r^2}\left(\frac{1}{\sin(\phi)}(\sin(\phi)u_\phi)_\phi + \frac{1}{\sin^2(\phi)}u_{\theta\theta}\right),
\end{aligned}$$

which can also be found in Section A.5 of the textbook, and we will prove it in our solution to Exercise 9.4 below. In particular, the radial part of the Laplacian is

$$u_{rr} + \frac{2}{r}u_r.$$

Therefore, the Cauchy problem

$$\begin{aligned}
u_{tt} - c^2\Delta u = 0 &\qquad x \in \mathbb{R}^3, t > 0, \\
u(x,0) = f(x) &\quad x \in \mathbb{R}^3, \\
u_t(x,0) = g(x) &\quad x \in \mathbb{R}^3.
\end{aligned}$$

is transformed into

$$\begin{aligned}
u_{tt} - c^2\left(u_{rr} + \frac{2}{r}u_r\right) = 0 &\qquad r \geq 0, t > 0, \\
u(r,0) = f(r) &\quad r \geq 0, \\
u_t(r,0) = g(r) &\quad r \geq 0,
\end{aligned}$$

and we do not need to include the angular part of the Laplacian because we are finding a radially symmetric solution. Now substitute $v(r,t) := ru(r,t)$. Then we obtain the first partial derivatives

$$\begin{aligned}
v_t &= ru_t, \\
v_r &= u + ru_r
\end{aligned}$$

and the second partial derivatives

$$\begin{aligned}
v_{tt} &= ru_{tt}, \\
v_{rr} &= ru_{rr} + \frac{2}{r}u_r.
\end{aligned}$$

Consequently, the partial differential equation implies

$$\begin{aligned}
0 &= u_{tt} - c^2\left(u_{rr} + \frac{2}{r}u_r\right) \\
&= \frac{1}{r}ru_{tt} - \frac{c^2}{r}(ru_{rr} + 2u_r) \\
&= \frac{1}{r}v_{tt} - \frac{c^2}{r}v_{rr} \\
&= \frac{1}{r}(v_{tt} - c^2v_{rr}),
\end{aligned}$$

from which we deduce the one-dimensional wave equation

$$v_{tt} - c^2v_{rr} = 0.$$

So the Cauchy problem is once again transformed into

$$\begin{aligned}
v_{tt} - c^2v_{rr} = 0 &\qquad r \geq 0, t > 0, \\
v(r,0) = rf(r) &\quad r \geq 0, \\
v_t(r,0) = rg(r) &\quad r \geq 0.
\end{aligned}$$

The solution of this problem is d'Alembert's formula

$$v(r,t) = \frac{1}{2}((r+ct)f(r+ct) + (r-ct)f(x-ct)) + \frac{1}{2c}\int_{r-ct}^{r+ct} sg(s)\,ds.$$

Therefore, the formal radial solution is

$$\begin{aligned}
u(r,t) &= \frac{1}{r}v(r,t) \\
&= \frac{1}{r}\left(\frac{1}{2}((r+ct)f(r+ct)+(r-ct)f(x-ct)) + \frac{1}{2c}\int_{r-ct}^{r+ct} sg(s)\,ds\right) \\
&= \frac{1}{2r}((r+ct)f(r+ct)+(r-ct)f(x-ct)) + \frac{1}{2cr}\int_{r-ct}^{r+ct} sg(s)\,ds.
\end{aligned}$$

Finally, we note that this $u(r,t)$ is only defined on the ray $r \geq 0$. But if we want the solution to be defined on $-\infty < r < \infty$, we need to write instead

$$u(r,t) = \frac{1}{2r}((r+ct)\tilde{f}(r+ct)+(r-ct)\tilde{f}(x-ct)) + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\tilde{g}(s)\,ds,$$

where $\tilde{f}$ and $\tilde{g}$ are the even extensions of f and g, respectively. □

9.4. Derive the formulation of the Laplace equation in a spherical coordinate system (r, θ, ϕ).

Solution - use polar coordinates. We know already that the Laplacian is defined in the Cartesian coordinate system by

$$\Delta u = u_{xx} + u_{yy} + u_{zz}.$$

To compute the Laplace equation $\Delta u = 0$ in the spherical coordinate system, we need to derive the equivalent expression of the Laplacian in spherical coordinates. Let

$$\begin{aligned}
x &= x(r,\theta,\phi) = r\sin(\phi)\cos(\theta), \\
y &= y(r,\theta,\phi) = r\sin(\phi)\sin(\theta), \\
z &= z(r,\theta,\phi) = r\cos(\phi), \\
u(x,y,z) &= w(r,\theta,\phi) = u(x_1(r,\theta,\phi), x_2(r,\theta,\phi), x_3(r,\theta,\phi)).
\end{aligned}$$

Set $s := \sqrt{x^2+y^2}$. Then we have

$$\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} = \sqrt{s^2+z^2}, \\
s &= \sqrt{x^2+y^2} = r\sin(\phi), \\
x &= r\sin(\phi)\cos(\theta) = s\cos(\theta), \\
y &= r\sin(\phi)\sin(\theta) = s\sin(\theta).
\end{aligned}$$

We know already from Exercise 7.7(a) that the Laplacian defined in the polar coordinate systems (s, θ) and (r, ϕ) are given by

$$\begin{aligned}
u_{xx} + u_{yy} &= w_{ss} + \frac{1}{s}w_s + \frac{1}{s^2}w_{\theta\theta}, \\
u_{ss} + u_{zz} &= w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\phi\phi},
\end{aligned}$$

respectively. So the three-dimensional Laplacian can be written

$$\begin{aligned}
\Delta u &= u_{xx} + u_{yy} + u_{zz} \\
&= \left(\cancel{w_{ss}} + \frac{1}{s}w_s + \frac{1}{s^2}w_{\theta\theta}\right) + \left(w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\phi\phi} - \cancel{w_{ss}}\right) \\
&= w_{rr} + \frac{1}{r}w_r + \frac{1}{s}w_s + \frac{1}{s^2}w_{\theta\theta} + \frac{1}{r^2}w_{\phi\phi}.
\end{aligned}$$

Our remaining work is to find w_s. From $r = \sqrt{s^2+z^2}$, we obtain the first partial derivative

$$\begin{aligned}
r_s &= (\sqrt{s^2+z^2})_s \\
&= \frac{s}{\sqrt{s^2+z^2}} \\
&= \frac{r\sin(\phi)}{r} \\
&= \sin(\phi).
\end{aligned}$$

Next, we see that, on the s-axis, we have $y = mx$, where m is the slope of y, and so we obtain

$$\begin{aligned}\tan(\theta) &= \frac{y}{x} \\ &= \frac{mx}{x} \\ &= m,\end{aligned}$$

and we can differentiate with respect to s both sides to further obtain

$$\begin{aligned}\sec^2(\theta)\theta_s &= (\tan(\theta))_s \\ &= m_s \\ &= 0,\end{aligned}$$

which implies the first partial derivative

$$\theta_s = 0$$

because $0 \leq \cos^2(\theta) \leq 1$ implies $\sec^2(\theta) \geq 1 > 0$. Finally, from $s = r\sin(\phi)$, we obtain

$$\begin{aligned}1 &= (s)_s \\ &= (r\sin(\phi))_s \\ &= r_s \sin(\phi) + r\cos(\phi)\phi_s \\ &= \sin(\phi)\sin(\phi) + r\cos(\phi)\phi_s \\ &= \sin^2(\phi) + r\cos(\phi)\phi_s,\end{aligned}$$

which implies the first partial derivative

$$\begin{aligned}\phi_s &= \frac{1 - \sin^2(\phi)}{r\cos(\phi)} \\ &= \frac{\cos^2(\phi)}{r\cos(\phi)} \\ &= \frac{\cos(\phi)}{r}.\end{aligned}$$

Therefore, by the multivariable chain rule, we have

$$\begin{aligned}w_s &= w_r r_s + w_\theta \theta_s + w_\phi \phi_s \\ &= w_r \sin(\phi) + w_\theta \cdot 0 + w_\phi \frac{\cos(\phi)}{r} \\ &= (\sin(\phi))w_r + \frac{\cos(\phi)}{r} w_\phi.\end{aligned}$$

Therefore, the Laplacian in spherical coordinates is

$$\begin{aligned}\Delta w &= w_{rr} + \frac{1}{r}w_r + \frac{1}{s}w_s + \frac{1}{s^2}w_{\theta\theta} + \frac{1}{r^2}w_{\phi\phi} \\ &= w_{rr} + \frac{1}{r}w_r + \frac{1}{r\sin(\phi)}\left((\sin(\phi))w_r + \frac{\cos(\phi)}{r}w_\phi\right) + \frac{1}{r^2\sin^2(\phi)}w_{\theta\theta} + \frac{1}{r^2}w_{\phi\phi} \\ &= w_{rr} + \frac{1}{r}w_r + \left(\frac{1}{r}w_r + \frac{1}{r^2}\frac{\cos(\phi)}{\sin(\phi)}w_\phi\right) + \frac{1}{r^2\sin^2(\phi)}w_{\theta\theta} + \frac{1}{r^2}w_{\phi\phi} \\ &= w_{rr} + \frac{2}{r}w_r + \frac{1}{r^2}\left(w_{\phi\phi} + \frac{\cos(\phi)}{\sin(\phi)}w_\phi + \frac{1}{\sin^2(\phi)}w_{\theta\theta}\right) \\ &= \frac{1}{r^2}(r^2 w_{rr} + 2rw_r) + \frac{1}{r^2}\left(\frac{1}{\sin(\phi)}(\sin(\phi)w_{\phi\phi} + \cos(\phi)w_\phi) + \frac{1}{\sin^2(\phi)}w_{\theta\theta}\right) \\ &= \frac{1}{r^2}(r^2 w_r)_r + \frac{1}{r^2}\left(\frac{1}{\sin(\phi)}(\sin(\phi)w_\phi)_\phi + \frac{1}{\sin^2(\phi)}w_{\theta\theta}\right),\end{aligned}$$

which agrees with the last one of the five formulas written in Section A.5 of the textbook. This means that the Laplace equation $\Delta u = 0$ in spherical coordinates is written

$$\frac{1}{r^2}(r^2 w_r)_r + \frac{1}{r^2}\left(\frac{1}{\sin(\phi)}(\sin(\phi)w_\phi)_\phi + \frac{1}{\sin^2(\phi)}w_{\theta\theta}\right) = 0,$$

as desired. □

Alternate solution - use Cartesian coordinates. We know already that the Laplacian is defined in the Cartesian coordinate system by

$$\Delta u = u_{xx} + u_{yy} + u_{zz}.$$

To compute the Laplace equation $\Delta u = 0$ in the spherical coordinate system, we need to derive the equivalent expression of the Laplacian in spherical coordinates. Let

$$\begin{aligned} x &= x(r,\theta,\phi) = r\sin(\phi)\cos(\theta), \\ y &= y(r,\theta,\phi) = r\sin(\phi)\sin(\theta), \\ z &= z(r,\theta,\phi) = r\cos(\phi), \\ u(x,y,z) &= w(r,\theta,\phi) = u(x_1(r,\theta,\phi), x_2(r,\theta,\phi), x_3(r,\theta,\phi)), \end{aligned}$$

the first three of which imply

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2}, \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right), \\ \phi &= \cos^{-1}\left(\frac{z}{r}\right). \end{aligned}$$

We obtain first partial derivatives

$$\begin{aligned} r_x &= (\sqrt{x^2+y^2+z^2})_x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}, \\ r_y &= (\sqrt{x^2+y^2+z^2})_x = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r}, \\ r_z &= (\sqrt{x^2+y^2+z^2})_z = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r}, \\ \theta_x &= \left(\tan^{-1}\left(\frac{y}{x}\right)\right)_x = -\frac{y}{x^2+y^2}, \\ \theta_y &= \left(\tan^{-1}\left(\frac{y}{x}\right)\right)_y = \frac{x}{x^2+y^2}, \\ \theta_z &= \left(\tan^{-1}\left(\frac{y}{x}\right)\right)_z = 0, \\ \phi_x &= \left(\cos^{-1}\left(\frac{z}{r}\right)\right)_x = \frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}, \\ \phi_y &= \left(\cos^{-1}\left(\frac{z}{r}\right)\right)_y = \frac{yz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}, \\ \phi_z &= \left(\cos^{-1}\left(\frac{z}{r}\right)\right)_z = -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}, \end{aligned}$$

and the second partial derivatives

$$\begin{aligned} r_{xx} &= \left(\frac{x}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}\right)_x = \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{y^2+z^2}{r^3}, \\ r_{yy} &= \left(\frac{y}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}\right)_y = \frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{x^2+z^2}{r^3}, \\ r_{zz} &= \left(\frac{z}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}\right)_z = \frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{x^2+y^2}{r^3}, \\ \theta_{xx} &= \left(-\frac{y}{x^2+y^2}\right)_x = \frac{2xy}{(x^2+y^2)^2}, \\ \theta_{yy} &= \left(\frac{x}{x^2+y^2}\right)_y = -\frac{2xy}{(x^2+y^2)^2}, \\ \theta_{zz} &= (0)_z = 0, \\ \phi_{xx} &= \left(\frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}\right)_x = z\frac{-2x^4+y^4-x^2y^2+y^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)^{\frac{3}{2}}}, \\ \phi_{yy} &= \left(\frac{yz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}\right)_y = z\frac{x^4-2y^4-x^2y^2+x^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)^{\frac{3}{2}}}, \\ \phi_{zz} &= \left(-\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}\right)_z = z\frac{2\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^2}. \end{aligned}$$

So, by the multivariable chain rule, we obtain the second partial derivatives

$$\begin{aligned}
u_{xx} &= (w(r,\theta,\phi))_{xx} \\
&= (w_r r_x + w_\theta \theta_x + w_\phi \phi_x)_x \\
&= (w_r r_x)_x + (w_\theta \theta_x)_x + (w_\phi \phi_x)_x \\
&= w_{rr}(r_x)^2 + w_r r_{xx} + w_{\theta\theta}(\theta_x)^2 + w_\theta \theta_{xx} + w_{\phi\phi}(\phi_x)^2 + w_\phi \phi_{xx} \\
&= w_{rr}\frac{x^2}{r^2} + w_r \frac{y^2+z^2}{r^3} + w_{\theta\theta}\frac{y^2}{(x^2+y^2)^2} + w_\theta \frac{2xy}{(x^2+y^2)^2} \\
&\quad + w_{\phi\phi}\frac{x^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)} + w_\phi z \frac{-2x^4+y^4-x^2y^2+y^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)^{\frac{3}{2}}}
\end{aligned}$$

and

$$\begin{aligned}
u_{yy} &= (w(r,\theta))_{yy} \\
&= (w_r r_y + w_\theta \theta_y + w_\phi \phi_x)_y \\
&= (w_r r_y)_y + (w_\theta \theta_y)_y + (w_\phi \phi_y)_y \\
&= w_{rr}(r_y)^2 + w_r r_{yy} + w_{\theta\theta}(\theta_y)^2 + w_\theta \theta_{yy} + w_{\phi\phi}(\phi_y)^2 + w_\phi \phi_{yy} \\
&= w_{rr}\frac{y^2}{r^2} + w_r \frac{x^2+z^2}{r^3} + w_{\theta\theta}\frac{x^2}{(x^2+y^2)^2} - w_\theta \frac{2xy}{(x^2+y^2)^2} \\
&\quad + w_{\phi\phi}\frac{y^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)} + w_\phi z \frac{x^4-2y^4-x^2y^2+x^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)^{\frac{3}{2}}}
\end{aligned}$$

and

$$\begin{aligned}
u_{zz} &= (w(r,\theta))_{zz} \\
&= (w_r r_z + w_\theta \theta_z + w_\phi \phi_x)_z \\
&= (w_r r_z)_z + (w_\theta \theta_z)_z + (w_\phi \phi_z)_z \\
&= w_{rr}(r_z)^2 + w_r r_{zz} + w_{\theta\theta}(\theta_z)^2 + w_\theta \theta_{zz} + w_{\phi\phi}(\phi_z)^2 + w_\phi \phi_{zz} \\
&= w_{rr}\frac{z^2}{r^2} + w_r \frac{x^2+y^2}{r^3} + w_{\theta\theta}\cdot 0 - 2w_\theta \cdot 0 + w_{\phi\phi}\frac{x^2+y^2}{(x^2+y^2+z^2)^2} + w_\phi z \frac{2\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^2} \\
&= w_{rr}\frac{z^2}{r^2} + w_r \frac{x^2+y^2}{r^3} + w_{\phi\phi}\frac{x^2+y^2}{(x^2+y^2+z^2)^2} + w_\phi z \frac{2\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^2}.
\end{aligned}$$

Therefore, the Laplacian in spherical coordinates is

$$
\begin{aligned}
\Delta u &= u_{xx} + u_{yy} + u_{zz} \\
&= \left(w_{rr}\frac{x^2}{r^2} + w_r \frac{y^2+z^2}{r^3} + w_{\theta\theta}\frac{y^2}{(x^2+y^2)^2} + \cancel{w_\theta \frac{2xy}{(x^2+y^2)^2}} \right. \\
&\qquad \left. + w_{\phi\phi}\frac{x^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)} + w_\phi z \frac{-2x^4+y^4-x^2y^2+y^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)^{\frac{3}{2}}} \right) \\
&\quad + \left(w_{rr}\frac{y^2}{r^2} + w_r\frac{x^2+z^2}{r^3} + w_{\theta\theta}\frac{x^2}{(x^2+y^2)^2} - \cancel{w_\theta\frac{2xy}{(x^2+y^2)^2}} + w_{\phi\phi}\frac{y^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)} \right. \\
&\qquad \left. + w_\phi z \frac{x^4-2y^4-x^2y^2+x^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)^{\frac{3}{2}}} \right) \\
&\quad + \left(w_{rr}\frac{z^2}{r^2} + w_r \frac{x^2+y^2}{r^3} + w_{\phi\phi}\frac{z^4}{(x^2+y^2+z^2)^2(x^2+y^2)} + w_\phi z \frac{2x^4+4x^2y^2+2y^4}{(x^2+y^2+z^2)^2(x^2+y^2)^{\frac{3}{2}}} \right) \\
&= w_{rr}\frac{x^2+y^2+z^2}{r^2} + w_r \frac{(y^2+z^2)+(x^2+z^2)+(x^2+y^2)}{r^3} + w_{\theta\theta}\frac{\cancel{x^2+y^2}}{(x^2+y^2)^{\cancel{2}1}} \\
&\quad + w_{\phi\phi}\left(\frac{x^2z^2+y^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)} + \frac{x^2+y^2}{(x^2+y^2+z^2)^2} \right) \\
&\quad + w_\phi z \left(\frac{(-2x^4+y^4-x^2y^2+y^2z^2)+(x^4-2y^4-x^2y^2+x^2z^2)}{(x^2+y^2+z^2)^2(x^2+y^2)^{\frac{3}{2}}} + \frac{2\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^2} \right) \\
&= w_{rr}\frac{x^2+y^2+z^2}{r^2} + w_r\frac{2(x^2+y^2+z^2)}{r^3} + w_{\theta\theta}\frac{1}{x^2+y^2} + w_{\phi\phi}\frac{(x^2+y^2)z^2+(x^2+y^2)(x^2+y^2)}{(x^2+y^2+z^2)^2(x^2+y^2)} \\
&\quad + w_\phi z \frac{(-\cancel{2x^4}+y^4-\cancel{x^2y^2}+y^2z^2)+(x^4-\cancel{2y^4}-\cancel{x^2y^2}+x^2z^2)+(\cancel{2x^4}+\cancel{4}2x^2y^2+\cancel{2y^4})}{(x^2+y^2+z^2)^2(x^2+y^2)^{\frac{3}{2}}} \\
&= w_{rr}\frac{\cancel{r^2}}{\cancel{r^2}} + w_r\frac{2\cancel{r^2}}{r^{\cancel{3}1}} + w_{\theta\theta}\frac{1}{x^2+y^2} + w_{\phi\phi}\frac{\cancel{(x^2+y^2+z^2)}\cancel{(x^2+y^2)}}{(x^2+y^2+z^2)^{\cancel{2}1}\cancel{(x^2+y^2)}} + w_\phi z\frac{x^4+y^4+2x^2y^2+x^2z^2+y^2z^2}{(x^2+y^2+z^2)^2(x^2+y^2)^{\frac{3}{2}}} \\
&= w_{rr} + \frac{2}{r}w_r + w_{\theta\theta}\frac{1}{x^2+y^2} + w_{\phi\phi}\frac{1}{x^2+y^2+z^2} + w_\phi z \frac{\cancel{(x^2+y^2+z^2)}\cancel{(x^2+y^2)}}{(x^2+y^2+z^2)^{\cancel{2}1}(x^2+y^2)^{\cancel{\frac{3}{2}}\frac{1}{2}}} \\
&= w_{rr} + \frac{2}{r}w_r + w_{\theta\theta}\frac{1}{r^2\sin^2(\phi)} + w_{\phi\phi}\frac{1}{r^2} + w_\phi \frac{z}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} \\
&= w_{rr} + \frac{2}{r}w_r + w_{\theta\theta}\frac{1}{r^2\sin^2(\phi)} + w_{\phi\phi}\frac{1}{r^2} + w_\phi\frac{\cancel{r}\cos(\phi)}{r^2(\cancel{r}\sin(\phi))} \\
&= w_{rr} + \frac{2}{r}w_r + \frac{1}{r^2}\left(\frac{1}{\sin^2(\phi)}w_{\theta\theta} + w_{\phi\phi} + \frac{\cos(\phi)}{\sin(\phi)}w_\phi \right) \\
&= \frac{1}{r^2}(r^2 w_{rr} + 2r w_r) + \frac{1}{r^2}\left(\frac{1}{\sin(\phi)}(\sin(\phi)w_{\phi\phi} + \cos(\phi)w_\phi) + \frac{1}{\sin^2(\phi)}w_{\theta\theta} \right) \\
&= \frac{1}{r^2}(r^2 w_r)_r + \frac{1}{r^2}\left(\frac{1}{\sin(\phi)}(\sin(\phi)w_\phi)_\phi + \frac{1}{\sin^2(\phi)}w_{\theta\theta} \right),
\end{aligned}
$$

which agrees with the last one of the five formulas written in Section A.5 of the textbook. This means that the Laplace equation $\Delta u = 0$ in spherical coordinates is written

$$\frac{1}{r^2}(r^2 w_r)_r + \frac{1}{r^2}\left(\frac{1}{\sin(\phi)}(\sin(\phi)w_\phi)_\phi + \frac{1}{\sin^2(\phi)}w_{\theta\theta} \right) = 0,$$

as desired. □

9.5. Find the radial solution to the Cauchy problem

$$
\begin{aligned}
u_{tt} - c^2\Delta u &= 0 \qquad && r \geq 0, t > 0, \\
u(r,0) &= 2 && r \geq 0, \\
u_t(r,0) &= 1 + r^2 && r \geq 0.
\end{aligned}
$$

Solution. We have already derived in Exercise 9.3 the radially symmetric solution

$$u(r,t) = \frac{1}{2r}\left((r+ct)\tilde{f}(r+ct) + (r-ct)\tilde{f}(r-ct)\right) + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\tilde{g}(s)\,ds. \tag{9.26}$$

Based on the given initial conditions, we have

$$f(r) = 2,$$
$$g(r) = 1 + r^2,$$

which are even functions. So their even extensions are

$$\tilde{f}(r) = f(r) = 2,$$
$$\tilde{g}(r) = g(r) = 1 + r^2.$$

Therefore, our radially symmetric solution is

$$\begin{aligned}
u(r,t) &= \frac{1}{2r}((r+ct)\tilde{f}(r+ct) + (r-ct)\tilde{f}(r-ct)) + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\tilde{g}(s)\,ds \\
&= \frac{1}{2r}((r+ct)2 + (r-ct)2) + \frac{1}{2cr}\int_{r-ct}^{r+ct} s(1+s^2)\,ds \\
&= 2 + \frac{1}{2cr}\int_{r-ct}^{r+ct} s + s^3\,ds \\
&= 2 + \frac{1}{2cr}\left(\left(\frac{1}{2}(r+ct)^2 + \frac{1}{4}(r+ct)^4\right) - \left(\frac{1}{2}(r-ct)^2 + \frac{1}{4}(r-ct)^4\right)\right) \\
&= 2 + \frac{1}{2cr}2crt(c^2t^2 + r^2 + 1) \\
&= \boxed{2 + t(c^2t^2 + r^2 + 1)},
\end{aligned}$$

as desired. □

9.6. Find the radial solution to the Cauchy problem

$$\begin{aligned}
u_{tt} - \Delta u &= 0 && r \geq 0, t > 0, \\
u(r,0) &= ae^{-r^2} && r \geq 0, \\
u_t(r,0) &= be^{-r^2} && r \geq 0,
\end{aligned}$$

where a and b are constants.

Solution. We have already derived in Exercise 9.3 the radially symmetric solution

$$u(r,t) = \frac{1}{2r}((r+ct)\tilde{f}(r+ct) + (r-ct)\tilde{f}(r-ct)) + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\tilde{g}(s)\,ds. \tag{9.26}$$

With $c = 1$, we have

$$u(r,t) = \frac{1}{2r}((r+t)\tilde{f}(r+t) + (r-t)\tilde{f}(r-t)) + \frac{1}{2r}\int_{r-t}^{r+t} s\tilde{g}(s)\,ds.$$

Based on the given initial conditions, we have

$$f(r) = ae^{-r^2},$$
$$g(r) = be^{-r^2},$$

which are even functions. So their even extensions are

$$\tilde{f}(r) = f(r) = ae^{-r^2},$$
$$\tilde{g}(r) = g(r) = be^{-r^2}.$$

Therefore, our radially symmetric solution is

$$\begin{aligned}
u(r,t) &= \frac{1}{2r}((r+t)\tilde{f}(r+t) + (r-t)\tilde{f}(r-t)) + \frac{1}{2r}\int_{r-t}^{r+t} s\tilde{g}(s)\,ds \\
&= \frac{1}{2r}((r+t)ae^{-(r+t)^2} + (r-t)ae^{-(r-t)^2}) + \frac{1}{2r}\int_{r-t}^{r+t} s(be^{-s^2})\,ds \\
&= \frac{1}{2r}((r+t)ae^{-(r+t)^2} + (r-t)ae^{-(r-t)^2}) + \frac{b}{2r}\int_{r-t}^{r+t} se^{-s^2}\,ds \\
&= \frac{1}{2r}((r+t)ae^{-(r+t)^2} + (r-t)ae^{-(r-t)^2}) + \frac{b}{2r}\left(-\frac{1}{2}e^{-(r+t)^2} + \frac{1}{2}e^{-(r-t)^2}\right) \\
&= \boxed{\frac{1}{4r}((2a(r+t) - b)e^{-(r+t)^2} + (2a(r-t) + b)e^{-(r-t)^2})},
\end{aligned}$$

as desired. □

9.7. Let h be a differentiable function in $\mathbb{R}^3$. We define its spherical mean $M_h(a)$ over the sphere of radius a around the point $\vec{x}$ to be

$$M_h(a,\vec{x}) := \frac{1}{4\pi a^2}\int_{|\vec{\xi}-\vec{x}|=a} h(\vec{\xi})\, ds_{\xi}. \tag{9.31}$$

Derive the Darboux equation

$$\left(\frac{\partial^2}{\partial a^2}+\frac{2}{a}\frac{\partial}{\partial a}\right) M_h(a,\vec{x}) = \Delta_x M_h(a,\vec{x}). \tag{9.32}$$

Solution. Consider the vectors $\vec{\xi} = (\xi_1,\xi_2,\xi_3), \vec{\eta} = (\eta_1,\eta_2,\eta_3) \in \mathbb{R}^3$. Apply the change of variable $\vec{\xi} := \vec{x}+a\vec{\eta}$, which implies $d\vec{\xi} = a^3\vec{d\eta}$ (keep in mind that a^3 is the Jacobian corresponding to the change of variable $\vec{\xi} := \vec{x}+a\vec{\eta}$) and $dS_{\xi} = a^2 dS_{\eta}$. Also, $|\vec{\xi}-\vec{x}| = a$ implies

$$\begin{aligned} a &= |\vec{\xi}-\vec{x}| \\ &= |\vec{\xi}-(\vec{\xi}+a\vec{\eta})| \\ &= |-a||\vec{\eta}| \\ &= a|\vec{\eta}|, \end{aligned}$$

which yields $|\vec{\eta}| = 1$ since a is positive. So (9.31) is equivalent to

$$M_h(a,\vec{x}) = \frac{1}{4\pi}\int_{|\vec{\eta}|=1} h(\vec{x}+a\vec{\eta})\, dS_{\eta}. \tag{9.35}$$

So we have the first partial derivative

$$\begin{aligned} \frac{\partial}{\partial a}M_h(a,\vec{x}) &= \frac{\partial}{\partial a}\left(\frac{1}{4\pi}\int_{|\vec{\eta}|=1} h(\vec{x}+a\vec{\eta})\, dS_{\eta}\right) \\ &= \frac{1}{4\pi}\frac{\partial}{\partial a}\left(\int_{|\vec{\eta}|=1} h(\vec{x}+a\vec{\eta})\, dS_{\eta}\right) \\ &= \frac{1}{4\pi}\int_{|\vec{\eta}|=1} \frac{\partial}{\partial a}h(\vec{x}+a\vec{\eta})\, dS_{\eta} \\ &= \frac{1}{4\pi}\int_{|\vec{\eta}|=1} \nabla h(\vec{x}+a\vec{\eta})\cdot\frac{\partial}{\partial a}(a\vec{\eta})\, dS_{\eta} \\ &= \frac{1}{4\pi}\int_{|\vec{\eta}|=1} \nabla h(\vec{x}+a\vec{\eta})\cdot\vec{\eta}\, dS_{\eta}. \end{aligned}$$

We recall the Divergence Theorem, which states

$$\int_D \nabla\cdot\vec{\psi}(\vec{\xi})\, d\vec{\xi} = \int_{\partial D}\vec{\psi}(\vec{\xi})\cdot\hat{n}\, dS$$

for any domain $D \subset \mathbb{R}^3$. Applying the Divergence Theorem with $\xi := \vec{x}+a\vec{\eta}$, $\psi := \nabla h$, $\hat{n} := \vec{\eta}$ (because the unit radius vector $\vec{\eta}$ is orthogonal to the sphere $\partial D = \{\vec{\eta}\in\mathbb{R}^3 \mid |\vec{\eta}| = 1\}$), and $D := \{\vec{\eta}\in\mathbb{R}^3 \mid |\vec{\eta}| < 1\}$, we have

$$\int_{|\vec{\eta}|<1} \nabla\cdot\nabla h(\vec{x}+a\vec{\eta})\, d(\vec{x}+a\vec{\eta}) = \int_{|\vec{\eta}|=1}\nabla h(\vec{x}+a\vec{\eta})\cdot\vec{\eta}\, dS_{\eta}.$$

Therefore, we have

$$\begin{aligned} \frac{\partial}{\partial a}M_h(a,\vec{x}) &= \frac{1}{4\pi}\int_{|\vec{\eta}|=1}\nabla h(\vec{x}+a\vec{\eta})\cdot\vec{\eta}\, dS_{\eta} \\ &= \frac{1}{4\pi}\int_{|\vec{\eta}|<1}\nabla\cdot\nabla h(\vec{x}+a\vec{\eta})\, d(\vec{x}+a\vec{\eta}) \\ &= \frac{1}{4\pi}\int_{|\vec{\eta}|<1}\Delta_x h(\vec{x}+a\vec{\eta})a\, d\vec{\eta} \\ &= \frac{1}{4\pi}\int_{|\vec{\xi}-\vec{x}|<a}\Delta_x h(\vec{\xi})a\,\frac{d\vec{\xi}}{a^3} \\ &= \frac{1}{4\pi a^2}\int_{|\vec{\xi}-\vec{x}|<a}\Delta_x h(\vec{\xi})\, d\vec{\xi} \\ &= \frac{1}{4\pi a^2}\Delta_x\left(\int_{|\vec{\xi}-\vec{x}|<a} h(\vec{\xi})\, d\vec{\xi}\right) \\ &= \frac{1}{4\pi a^2}\Delta_x\left(\int_0^a\int_{|\vec{\xi}-\vec{x}|=\alpha} h(\vec{\xi})\, d\vec{\xi}\, d\alpha\right), \end{aligned}$$

which is equivalent to, upon multiplying both sides by a^2,

$$a^2 \frac{\partial}{\partial a} M_h(a, \vec{x}) = \frac{1}{4\pi} \Delta_x \left(\int_0^a \int_{|\vec{\xi}-\vec{x}|=\alpha} h(\vec{\xi})\, d\vec{\xi}\, d\alpha \right).$$

Now, we can differentiate with respect to a both sides of the equation, writing

$$\frac{\partial}{\partial a} \left(a^2 \frac{\partial}{\partial a} M_h(a, \vec{x}) \right) = \frac{\partial}{\partial a} \left(\frac{1}{4\pi} \Delta_x \left(\int_0^a \int_{|\vec{\xi}-\vec{x}|=\alpha} h(\vec{\xi})\, d\vec{\xi}\, d\alpha \right) \right).$$

Using the product rule for derivatives, the left hand side is

$$\begin{aligned} \frac{\partial}{\partial a} \left(a^2 \frac{\partial}{\partial a} M_h(a, \vec{x}) \right) &= \frac{\partial}{\partial a}(a^2) \frac{\partial}{\partial a} M_h(a, \vec{x}) + a^2 \frac{\partial}{\partial a} \left(\frac{\partial}{\partial a} M_h(a, \vec{x}) \right) \\ &= 2a \frac{\partial}{\partial a} M_h(a, \vec{x}) + a^2 \frac{\partial^2}{\partial a^2} M_h(a, \vec{x}) \\ &= a^2 \left(\frac{\partial^2}{\partial a^2} + \frac{2}{a} \frac{\partial}{\partial a} \right) M_h(a, \vec{x}). \end{aligned}$$

Using the differential version of the Fundamental Theorem of Calculus, the right hand side is

$$\begin{aligned} \frac{\partial}{\partial a} \left(\frac{1}{4\pi} \Delta_x \left(\int_0^a \int_{|\vec{\xi}-\vec{x}|=\alpha} h(\vec{\xi})\, d\vec{\xi}\, d\alpha \right) \right) &= \frac{1}{4\pi} \frac{\partial}{\partial a} \Delta_x \left(\int_0^a \int_{|\vec{\xi}-\vec{x}|=\alpha} h(\vec{\xi})\, d\vec{\xi}\, d\alpha \right) \\ &= \frac{1}{4\pi} \Delta_x \frac{\partial}{\partial a} \left(\int_0^a \int_{|\vec{\xi}-\vec{x}|=\alpha} h(\vec{\xi})\, d\vec{\xi}\, d\alpha \right) \\ &= \frac{1}{4\pi} \Delta_x \left(\int_{|\vec{\xi}-\vec{x}|=a} h(\vec{\xi})\, d\vec{\xi} \right) \\ &= a^2 \Delta_x \left(\frac{1}{4\pi a^2} \int_{|\vec{\xi}-\vec{x}|=a} h(\vec{\xi})\, d\vec{\xi} \right) \\ &= a^2 \Delta_x M_h(a, \vec{x}). \end{aligned}$$

Equating both sides, we conclude

$$a^2 \left(\frac{\partial^2}{\partial a^2} + \frac{2}{a} \frac{\partial}{\partial a} \right) M_h(a, \vec{x}) = a^2 \Delta_x M_h(a, \vec{x}),$$

from which we can divide both sides by a^2 to obtain the Darboux equation

$$\left(\frac{\partial^2}{\partial a^2} + \frac{2}{a} \frac{\partial}{\partial a} \right) M_h(a, \vec{x}) = \Delta_x M_h(a, \vec{x}),$$

as desired. □