Solutions to assigned homework problems from *Fourier Analysis: An Introduction* by Elias Stein and Rami Sakarchi

Homework 1

- Sect. 1.3, pp. 23-27; 4, 5, 7, 9 (optional), 10.
- Sect. 1.4, p. 26: 1 (optional)
- Sect. 2.6, pp. 58-60: 1, 2, 4, 5, 6, 9

1.3.4. For $z \in \mathbb{C}$, we define the complex exponential by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(a) Prove that the above definition makes sense, by showing that the series converges for every complex number z. Moreover, show that the convergence is uniform on every bounded subset of $\mathbb{C}$.

Solution. To show that the series expansion of the complex exponential is convergent, we will use the ratio test from first-year calculus. Set $a_n(z) := \frac{z^n}{n!}$. Then we have the ratio

$$\begin{aligned}\left|\frac{a_{n+1}(z)}{a_n(z)}\right| &= \left|\frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}}\right| \\ &= \left|\frac{z^{n+1}}{(n+1)!}\frac{n!}{z^n}\right| \\ &= \frac{|z|}{(n+1)},\end{aligned}$$

which implies its limit

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}(z)}{a_n(z)}\right| &= \lim_{n\to\infty}\frac{|z|}{(n+1)} \\ &= 0.\end{aligned}$$

So the series is convergent.

Next, we will show that converge is uniform on every bounded subset of $\mathbb{C}$. Every bounded subset of $\mathbb{C}$ is contained in the disk $D_R := \{z \in \mathbb{C} \mid |z| < R\}$. Let $S_n(z) := \sum_{k=0}^{n} \frac{z^k}{k!}$ and $S(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$. Then our goal is to show that $S_k(z)$ converges pointwise to $S(z)$. Observe that factorials grow faster than exponentials; that is, given $R > 0$ we have $k! > (2R)^k$ for all large positive integers k. One can justify that using a proof by induction. Observe that, if we never assumed that the subset of $\mathbb{C}$ is not bounded, as seen in part (a), then the disk containing the unbounded subset must have an infinite radius, i.e. $R = \infty$, which is all of $\mathbb{C}$. But this implies $k! > (2R)^k = \infty$, which would be a contradiction because in reality we have $k! < \infty$ for all positive integers k. With all that said, for any integers m, n with the assumption

$m > n$ without loss of generality, we use the triangle inequality and the geometric partial sum formula to obtain

$$
\begin{aligned}
|S_m(z) - S_n(z)| &= \left| \sum_{k=0}^{m} \frac{z^k}{k!} - \sum_{k=0}^{n} \frac{z^k}{k!} \right| \\
&= \left| \sum_{k=n+1}^{m} \frac{z^k}{k!} \right| \\
&\leq \sum_{k=n+1}^{m} \frac{|z|^k}{k!} \\
&< \sum_{k=n+1}^{m} \frac{R^k}{k!} \\
&< \sum_{k=n+1}^{m} \frac{R^k}{(2R)^k} \\
&= \sum_{k=n+1}^{m} \frac{1}{2^k} \\
&= \sum_{k=0}^{m} \frac{1}{2^k} - \sum_{k=0}^{n} \frac{1}{2^k} \\
&= \frac{1 - (\frac{1}{2})^{m+1}}{1 - \frac{1}{2}} - \frac{1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} \\
&= \frac{(\frac{1}{2})^{n+1} - (\frac{1}{2})^{m+1}}{\frac{1}{2}} \\
&= \frac{1}{2^n} - \frac{1}{2^m}
\end{aligned}
$$

Send $m \to \infty$ of our result to conclude

$$
\begin{aligned}
|S_n(z) - S(z)| &= |S_n(z) - \lim_{m\to\infty} S_m(z)| \\
&= \lim_{m\to\infty} |S_n(z) - S_m(z)| \\
&< \lim_{m\to\infty} \left(\frac{1}{2^n} - \frac{1}{2^m} \right) \\
&= \frac{1}{2^n} \\
&< \frac{1}{n},
\end{aligned}
$$

where we used $2^n > n$ for all positive integers n (one can prove this using a proof by induction). Finally, let $\epsilon > 0$ be given, and choose $N > \frac{1}{\epsilon}$. If $n \geq N$, then we have

$$
\begin{aligned}
|S_n(z) - S(z)| &< \frac{1}{n} \\
&\leq \frac{1}{N} \\
&< \epsilon
\end{aligned}
$$

for all $z \in D_R$. Therefore, $S_n(z)$ converges uniformly to $S(z)$ on D_R, which implies the same result on any bounded set of $\mathbb{C}$. □

(b) If z_1, z_2 are two complex numbers, prove that $e^{z_1}e^{z_2} = e^{z_1+z_2}$. [Hint: Use the binomial theorem to expand $(z_1 + z_2)^n$, as well as the formula for the binomial coefficients.]

Solution. Using the binomial theorem

$$
(z_1 + z_2)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k}.
$$

and the power series expansion

$$
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},
$$

we obtain

$$
\begin{aligned}
e^{z_1+z_2} &= \sum_{n=0}^{\infty} \frac{(z_1+z_2)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z_1^k z_2^{n-k}}{k!(n-k)!} \\
&= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{z_1^l z_2^m}{l!m!} \\
&= \left(\sum_{l=0}^{\infty} \frac{z_1^l}{l!}\right)\left(\sum_{m=0}^{\infty} \frac{z_1^m}{m!}\right) \\
&= e^{z_1} e^{z_2},
\end{aligned}
$$

as desired. For complete clarity of our proof, we will need to justify the Cauchy product

$$\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{z_1^l z_2^m}{l!m!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z_1^k z_2^{n-k}}{k!(n-k)!}$$

Credit for the following goes to this question on Mathematics StackExchange. The double sum of the left-hand side

$$\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{z_1^l z_2^m}{l!m!} = \sum_{l=0}^{\infty} \underbrace{\left(\frac{z_1^l z_2^0}{l!0!} + \frac{z_1^l z_2^1}{l!1!} + \frac{z_1^l z_2^2}{l!2!} + \frac{z_1^l z_2^3}{l!3!} + \cdots\right)}_{\text{This is a sum of all terms along any row } l.} = \sum_{m=0}^{\infty} \underbrace{\left(\frac{z_1^0 z_2^m}{0!m!} + \frac{z_1^1 z_2^m}{1!m!} + \frac{z_1^2 z_2^m}{2!m!} + \frac{z_1^3 z_2^m}{3!m!} + \cdots\right)}_{\text{This is a sum of all terms along any column } m.}$$

runs down all the rows OR all the columns of the following array:

$$
\begin{array}{c|ccccccccc}
 & & m=0 & & m=1 & & m=2 & & m=3 & \cdots \\
\hline
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
l=0 & \rightarrow & \frac{z_1^0 z_2^0}{0!0!} & \rightarrow & \frac{z_1^0 z_2^1}{0!1!} & \rightarrow & \frac{z_1^0 z_2^2}{0!2!} & \rightarrow & \frac{z_1^0 z_2^3}{0!3!} & \rightarrow \ \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
l=1 & \rightarrow & \frac{z_1^1 z_2^0}{1!0!} & \rightarrow & \frac{z_1^1 z_2^1}{1!1!} & \rightarrow & \frac{z_1^1 z_2^2}{1!2!} & \rightarrow & \frac{z_1^1 z_3^2}{1!3!} & \rightarrow \ \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
l=2 & \rightarrow & \frac{z_1^2 z_2^0}{2!0!} & \rightarrow & \frac{z_1^2 z_2^1}{2!1!} & \rightarrow & \frac{z_1^2 z_2^2}{2!2!} & \rightarrow & \frac{z_1^2 z_2^3}{2!3!} & \rightarrow \ \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
l=3 & \rightarrow & \frac{z_1^3 z_2^0}{3!0!} & \rightarrow & \frac{z_1^3 z_2^1}{3!1!} & \rightarrow & \frac{z_1^3 z_2^2}{3!2!} & \rightarrow & \frac{z_1^3 z_2^3}{3!3!} & \rightarrow \ \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \ddots
\end{array}
$$

Similarly, the double sum of the right-hand side

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z_1^k z_2^{n-k}}{k!(n-k)!} = \sum_{n=0}^{\infty} \underbrace{\left(\frac{z_1^0 z_2^{n-0}}{0!(n-0)!} + \frac{z_1^1 z_2^{n-1}}{1!(n-1)!} + \frac{z_1^2 z_2^{n-2}}{2!(n-2)!} + \frac{z_1^3 z_2^{n-3}}{3!(n-3)!} + \cdots + \frac{z_1^n z_2^{n-n}}{n!(n-n)!}\right)}_{\text{This is a sum of all terms along any antidiagonal } n.}$$

runs down all the antidiagonals of the following array:

$$
\begin{array}{c|cccccccccc}
 & & & & n=0 & & n=1 & & n=2 & & n=3 \ \cdots \\
\hline
 & & & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\
k=0 & \rightarrow & \frac{z_1^0 z_2^{0-0}}{0!(0-0)!} & \rightarrow & \frac{z_1^0 z_2^{1-0}}{0!(1-0)!} & \rightarrow & \frac{z_1^0 z_2^{2-0}}{0!(2-0)!} & \rightarrow & \frac{z_1^0 z_2^{3-0}}{0!(3-0)!} & \rightarrow & \cdots \\
 & & & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\
k=1 & \rightarrow & \frac{z_1^1 z_2^{1-1}}{1!(1-1)!} & \rightarrow & \frac{z_1^1 z_2^{2-1}}{1!(2-1)!} & \rightarrow & \frac{z_1^1 z_2^{3-1}}{1!(3-1)!} & \rightarrow & \frac{z_1^1 z_2^{4-1}}{1!(4-1)!} & \rightarrow & \cdots \\
 & & & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\
k=2 & \rightarrow & \frac{z_1^2 z_2^{2-2}}{2!(2-2)!} & \rightarrow & \frac{z_1^2 z_2^{3-2}}{2!(3-2)!} & \rightarrow & \frac{z_1^2 z_2^{4-2}}{2!(4-2)!} & \rightarrow & \frac{z_1^2 z_2^{5-2}}{2!(5-2)!} & \rightarrow & \cdots \\
 & & & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\
k=3 & \rightarrow & \frac{z_1^3 z_2^{3-3}}{3!(3-3)!} & \rightarrow & \frac{z_1^3 z_2^{4-3}}{3!(4-3)!} & \rightarrow & \frac{z_1^3 z_2^{5-3}}{3!(5-3)!} & \rightarrow & \frac{z_1^3 z_2^{6-2}}{3!(6-3)!} & \rightarrow & \cdots \\
 & & & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}
$$

Notice that the **corresponding entries** of our two arrays are equal to each other. This implies that our double-sums of the left- and right-hand sides are equal, as we claimed, which completes our justification of the Cauchy product. □

(c) Show that if z is purely imaginary, that is, $z = iy$ with $y \in \mathbb{R}$, then $e^{iy} = \cos(y) + i\sin(y)$. This is Euler's identity. [Hint: Use power series.]

Solution. Using power series expansions

$$\begin{aligned} e^{iy} &= \sum_{n=0}^{\infty} \frac{(iy)^n}{n!}, \\ \cos(y) &= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!}, \\ \sin(y) &= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!}, \end{aligned}$$

we obtain

$$\begin{aligned} e^{iy} &= \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(iy)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(iy)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(i^2)^n y^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(i^2)^n y^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} \\ &= \cos(y) + i\sin(y), \end{aligned}$$

as desired. □

(d) More generally,

$$e^{x+iy} = e^x(\cos(y) + i\sin(y))$$

whenever $x, y \in \mathbb{R}$, and show that

$$|e^{x+iy}| = e^x.$$

Solution. We have

$$\begin{aligned} e^{x+iy} &= e^x e^{iy} \\ &= e^x(\cos(y) + i\sin(y)) \end{aligned}$$

and

$$\begin{aligned} |e^{x+iy}| &= |e^x e^{iy}| \\ &= |e^x(\cos(y) + i\sin(y))| \\ &= |e^x||\cos(y) + i\sin(y)| \\ &= e^x\sqrt{\cos^2(y) + \sin^2(y)} \\ &= e^x \cdot 1 \\ &= e^x, \end{aligned}$$

as desired. □

(e) Prove that $e^z = 1$ if and only if $z = 2\pi ki$ for some integer k.

Solution. We can always write $z = x + iy$, which means

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x(\cos(y) + i\sin(y)) \\ &= e^x\cos(y) + ie^x\sin(y). \end{aligned}$$

This will be useful in proving our following implications.
Assume $e^z = 1 = 1 + i0$. Then we have

$$1 + i0 = e^x\cos(y) + ie^x\sin(y).$$

Equate the real and imaginary components of a complex number to conclude the system of equations

$$e^x \cos(y) = 1,$$
$$e^x \sin(y) = 0.$$

Since we know $e^x > 0$ for all $x \in \mathbb{R}$, we must have $\sin(y) = 0$, or equivalently $y = 2\pi k$ for any integer k. Furthermore, knowing $\cos(2\pi k) = 1$, we also obtain

$$\begin{aligned} e^x &= e^x \cos(2\pi k) \\ &= e^x \cos(y) \\ &= 1 \end{aligned}$$

has the solution $x = 0$ only. We conclude

$$\begin{aligned} z &= x + iy \\ &= 0 + 2\pi k i \\ &= 2\pi k i, \end{aligned}$$

as desired.

Conversely, assume $z = 2\pi k i$. Then by Euler's formula we have

$$\begin{aligned} e^z &= e^{2\pi k i} \\ &= \cos(2\pi k) + i \sin(2\pi k) \\ &= 1 + i0 \\ &= 1, \end{aligned}$$

as desired. □

(f) Show that every complex number $z = x + iy$ can be written in the form $z = re^{i\theta}$, where r is unique and in the range $0 \le r < \infty$, and $\theta \in \mathbb{R}$ is unique up to an integer multiple of 2π. Check that

$$r = |z| \text{ and } \theta = \arctan\left(\frac{y}{x}\right)$$

whenever these formulas make sense.

Solution. We can write $x = r\cos(\theta)$ and $y = r\sin(\theta)$. By Euler's formula, we have

$$\begin{aligned} z &= x + iy \\ &= r\cos(\theta) + ir\sin(\theta) \\ &= r(\cos(\theta) + i\sin(\theta)) \\ &= re^{i\theta}, \end{aligned}$$

as desired. □

(g) In particular, $i = e^{\frac{i\pi}{2}}$. What is the geometric meaning of multiplying a complex number by i? Or by $e^{i\theta}$ for any $\theta \in \mathbb{R}$?

Solution. We have

$$\begin{aligned} e^{i\frac{\pi}{2}} &= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) \\ &= 0 + i \cdot 1 \\ &= i, \end{aligned}$$

as desired. Multiplying a complex number by $e^{i\theta}$ rotates the point along a circle in $\mathbb{C}$. □

(h) Given $\theta \in \mathbb{R}$, show that

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

These are also called Euler's identities.

Solution. By Euler's formula, we have

$$\begin{aligned} \frac{e^{i\theta} + e^{-i\theta}}{2} &= \frac{(\cos(\theta) + i\sin(\theta)) + (\cos(-\theta) + i\sin(-\theta))}{2} \\ &= \frac{(\cos(\theta) + i\sin(\theta)) + (\cos(\theta) - i\sin(\theta))}{2} \\ &= \frac{2\cos(\theta)}{2} \\ &= \cos(\theta) \end{aligned}$$

and

$$
\begin{aligned}
\frac{e^{i\theta} - e^{-i\theta}}{2i} &= \frac{(\cos(\theta) + i\sin(\theta)) - (\cos(-\theta) + i\sin(-\theta))}{2} \\
&= \frac{(\cos(\theta) + i\sin(\theta)) - (\cos(\theta) - i\sin(\theta))}{2i} \\
&= \frac{2i\sin(\theta)}{2i} \\
&= \sin(\theta),
\end{aligned}
$$

as desired. □

(i) Use the complex exponential to derive trigonometric identities such as

$$\cos(\theta + \vartheta) = \cos(\theta)\cos(\vartheta) - \sin(\theta)\sin(\vartheta),$$

and then show that

$$
\begin{aligned}
2\sin(\theta)\sin(\varphi) &= \cos(\theta - \varphi) - \cos(\theta + \varphi), \\
2\sin(\theta)\cos(\varphi) &= \sin(\theta + \varphi) + \sin(\theta - \varphi).
\end{aligned}
$$

This calculation connects the solution given by d'Alembert in terms of traveling waves and the solution in terms of superposition of standing waves.

Solution. We have

$$
\begin{aligned}
\cos(\theta + \vartheta) + i\sin(\theta + \vartheta) &= e^{i(\theta + \vartheta)} \\
&= e^{i\theta}e^{i\vartheta} \\
&= (\cos(\theta) + i\sin(\theta))(\cos(\vartheta) + i\sin(\vartheta)) \\
&= \cos(\theta)\cos(\vartheta) - \sin(\theta)\sin(\vartheta) + i(\cos(\theta)\sin(\vartheta) + \sin(\theta)\cos(\vartheta)),
\end{aligned}
$$

from which we can equate the real and imaginary components to obtain

$$
\begin{aligned}
\cos(\theta + \vartheta) &= \cos(\theta)\cos(\vartheta) - \sin(\theta)\sin(\vartheta), \\
\sin(\theta + \vartheta) &= \cos(\theta)\sin(\vartheta) + \sin(\theta)\cos(\vartheta),
\end{aligned}
$$

respectively. Furthermore, we obtain

$$
\begin{aligned}
\sin(\theta + \varphi) + \sin(\theta - \varphi) &= (\cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)) + (\cos(\theta)\sin(-\varphi) + \sin(\theta)\cos(-\varphi)) \\
&= (\cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)) + (-\cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)) \\
&= 2\sin(\theta)\cos(\varphi)
\end{aligned}
$$

and

$$
\begin{aligned}
\cos(\theta - \varphi) - \cos(\theta + \varphi) &= (\cos(\theta)\cos(-\varphi) - \sin(\theta)\sin(-\varphi)) - (\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)) \\
&= (\cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi)) - (\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)) \\
&= 2\sin(\theta)\sin(\varphi),
\end{aligned}
$$

as desired. □

1.3.5. Verify that $f(x) = e^{inx}$ is periodic with period 2π and that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Use this fact to prove that if $n, m \geq 1$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

and similarly

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Finally, show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0$$

for any positive integers n, m. [Hint: Calculate $e^{inx}e^{-imx} + e^{inx}e^{imx}$ and $e^{inx}e^{-imx} - e^{inx}e^{imx}$.]

Solution. For all integers k, we have

$$\begin{aligned} f(x+2\pi k) &= e^{in(x+2\pi k)} \\ &= e^{inx}e^{2\pi ki} \\ &= f(x)\cdot 1 \\ &= f(x), \end{aligned}$$

which means f is periodic with period 2π. Next, we have, if $n \neq 0$,

$$\begin{aligned} \int_{-\pi}^{\pi} e^{inx}\,dx &= \left.\frac{e^{inx}}{in}\right|_{-\pi}^{\pi} \\ &= \frac{e^{in\pi}-e^{-in\pi}}{in} \\ &= \frac{(-1)^n-(-1)^n}{in} \\ &= \frac{0}{in} \\ &= 0 \end{aligned}$$

and, if $n = 0$,m

$$\begin{aligned} \int_{-\pi}^{\pi} e^{i(0)x}\,dx &= \int_{-\pi}^{\pi} 1\,dx \\ &= x|_{-\pi}^{\pi} \\ &= \pi-(-\pi) \\ &= 2\pi, \end{aligned}$$

thereby establishing

$$\int_{-\pi}^{\pi} e^{inx}\,dx = \begin{cases} 2\pi & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Now, following the given hint, we have

$$\begin{aligned} 2\cos(nx)\cos(mx) + i(2\sin(nx)\cos(mx)) &= 2(\cos(nx)+i\sin(nx))\cos(mx) \\ &= 2e^{inx}\frac{e^{imx}+e^{-imx}}{2} \\ &= e^{inx}(e^{imx}+e^{-imx}) \\ &= e^{inx}e^{-imx}+e^{inx}e^{imx} \\ &= e^{i(n-m)x}+e^{i(n+m)x} \end{aligned}$$

and

$$\begin{aligned} 2\sin(nx)\sin(mx) + i(-2\cos(nx)\sin(mx)) &= -2i(\cos(nx)+i\sin(nx))\sin(mx) \\ &= -2ie^{inx}\frac{e^{imx}-e^{-imx}}{2i} \\ &= e^{inx}(e^{-imx}-e^{imx}) \\ &= e^{inx}e^{-imx}-e^{inx}e^{imx} \\ &+ e^{i(n-m)x}-e^{i(n+m)x}. \end{aligned}$$

Add and subtract these two previous equations and divide both sides by 2 to obtain, respectively

$$\begin{aligned} (\cos(nx)\cos(mx)+\sin(nx)\sin(mx)) + i(\sin(nx)\cos(mx)-\cos(nx)\sin(mx)) &= e^{i(n-m)x}, \\ (\cos(nx)\cos(mx)-\sin(nx)\sin(mx)) + i(\sin(nx)\cos(mx)+\cos(nx)\sin(mx)) &= e^{i(n+m)x}. \end{aligned}$$

Add and subtract these two previous equations and divide both sides by 2 to obtain, respectively

$$\begin{aligned} \cos(nx)\cos(mx) + i\sin(nx)\cos(mx) &= \frac{e^{i(n-m)x}+e^{i(n+m)x}}{2}, \\ \sin(nx)\sin(mx) - i\sin(nx)\cos(mx) &= \frac{e^{i(n-m)x}-e^{i(n+m)x}}{2}. \end{aligned}$$

Integrate over $[-\pi, \pi]$ both sides of our latest two equations to conclude

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx + i\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = \begin{cases} 2\pi & \text{if } n-m=0, \\ 0 & \text{if } n-m\neq 0, \end{cases},$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx + i\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = \begin{cases} 2\pi & \text{if } n-m=0, \\ 0 & \text{if } n-m\neq 0, \end{cases}$$

from which we can equate the real and imaginary components of our last two equations to conclude simultaneously

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 1 & \text{if } n=m, \\ 0 & \text{if } n\neq m, \end{cases}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 1 & \text{if } n=m, \\ 0 & \text{if } n\neq m, \end{cases}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0,$$

as desired. □

1.3.7. Show that if a and b are real, then one can write

$$a\cos(ct) + b\sin(ct) = A\cos(ct-\varphi),$$

where $A = \sqrt{a^2+b^2}$, and φ is chosen so that

$$\cos(\varphi) = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \sin(\varphi) = \frac{b}{\sqrt{a^2+b^2}}.$$

Solution. Note that with the notation given in the problem we can also write

$$\cos(\varphi) = \frac{a}{A},$$
$$\sin(\varphi) = \frac{b}{A}.$$

We have

$$\begin{aligned}
A\cos(ct-\varphi) &= A\frac{e^{ct-\varphi} + e^{-(ct-\varphi)}}{2} \\
&= \frac{A}{2}(e^{ct}e^{-\varphi} + e^{-ct}e^{\varphi}) \\
&= \frac{A}{2}((\cos(ct) + i\sin(ct))(\cos(-\varphi) + i\sin(-\varphi)) \\
&\qquad + (\cos(-ct) + i\sin(-ct))(\cos(\varphi) + i\sin(\varphi))) \\
&= \frac{A}{2}((\cos(ct) + i\sin(ct))(\cos(\varphi) - i\sin(\varphi)) \\
&\qquad + (\cos(ct) - i\sin(ct))(\cos(\varphi) - i\sin(\varphi))) \\
&= \frac{A}{2}((\cos(ct)\cos(\varphi) + \sin(ct)\sin(\varphi)) + i(-\cos(ct)\sin(\varphi) + \sin(ct)\cos(\varphi)) \\
&\qquad + (\cos(ct)\cos(\varphi) + \sin(ct)\sin(\varphi)) + i(\cos(ct)\sin(\varphi) - \sin(ct)\cos(\varphi))) \\
&= \frac{A}{2}(2\cos(ct)\cos(\varphi) + 2\sin(ct)\sin(\varphi))) \\
&= A\cos(ct)\cos(\varphi) + A\sin(ct)\sin(\varphi) \\
&= A\cos(ct)\frac{a}{A} + A\sin(ct)\frac{b}{A} \\
&= a\cos(ct) + b\sin(ct),
\end{aligned}$$

as desired. □

1.3.9. In the case of the plucked string, use the formula for the Fourier sine coefficients to show that

$$A_m = \frac{2h}{m^2}\frac{\sin(mp)}{p(\pi-p)}.$$

For what position of p are the second, fourth, ... harmonics missing? For what position of p are the third, sixth, ... harmonics missing?

Solution. Page 17 of the Stein and Shakarchi textbook gives the function

$$f(x) = \begin{cases} \frac{xh}{p} & \text{if } 0 \leq x \leq p, \\ \frac{h(\pi - x)}{\pi - p} & \text{if } p \leq x \leq \pi, \end{cases}$$

which serves as a simplified model of a plucked string. Using the formula for the Fourier sine coefficients, we obtain

$$\begin{aligned} A_m &= \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)\,dx \\ &= \frac{2}{\pi}\left(\int_0^p f(x)\sin(mx)\,dx + \int_p^\pi f(x)\sin(mx)\,dx\right) \\ &= \frac{2}{\pi}\left(\int_0^p \frac{xh}{p}\sin(mx)\,dx + \int_p^\pi \frac{h(\pi - x)}{\pi - p}\sin(mx)\,dx\right) \\ &= \frac{2h}{\pi p}\int_0^p x\sin(mx)\,dx + \frac{2h}{\pi(\pi - p)}\int_p^\pi (\pi - x)\sin(mx)\,dx. \end{aligned}$$

for all positive integers m. We employ the method of integration by parts to obtain

$$\begin{aligned} \int_0^p x\sin(mx)\,dx &= -\frac{1}{m}x\cos(mx)\Big|_0^p + \frac{1}{m}\int_0^p \cos(mx)\,dx \\ &= -\frac{p\cos(mp) - 0\cos(m(0))}{m} + \frac{1}{m^2}\sin(mx)\Big|_0^p \\ &= -\frac{p\cos(mp)}{m} + \frac{\sin(mp) - \sin(m(0))}{m^2} \\ &= -\frac{p\cos(mp)}{m} + \frac{\sin(mp)}{m^2} \end{aligned}$$

and

$$\begin{aligned} \int_p^\pi (\pi - x)\sin(mx)\,dx &= -\frac{1}{m}(\pi - x)\cos(mx)\Big|_p^\pi - \frac{1}{m}\int_p^\pi \cos(mx)\,dx \\ &= -\frac{(\pi - \pi)\cos(m\pi) - (\pi - p)\cos(mp)}{m} - \frac{1}{m^2}\sin(mx)\Big|_p^\pi \\ &= -\frac{0\cos(m\pi) - (\pi - p)\cos(mp)}{m} - \frac{\sin(m\pi) - \sin(mp)}{m^2} \\ &= \frac{(\pi - p)\cos(mp)}{m} - \frac{0 - \sin(mp)}{m^2} \\ &= \frac{(\pi - p)\cos(mp)}{m} + \frac{\sin(mp)}{m^2}. \end{aligned}$$

So we have

$$\begin{aligned} A_m &= \frac{2h}{\pi p}\int_0^p x\sin(mx)\,dx + \frac{2h}{\pi(\pi - p)}\int_p^\pi (\pi - x)\sin(mx)\,dx \\ &= \frac{2h}{\pi p}\left(-\frac{p\cos(mp)}{m} + \frac{\sin(mp)}{m^2}\right) + \frac{2h}{\pi(\pi - p)}\left(\frac{(\pi - p)\cos(mp)}{m} + \frac{\sin(mp)}{m^2}\right) \\ &= -\frac{2h\cos(mp)}{\pi m} + \frac{2h\sin(mp)}{\pi m^2 p} + \frac{2h\cos(mp)}{\pi m} + \frac{2h\sin(mp)}{\pi m^2(\pi - p)} \\ &= \frac{2h\sin(mp)}{\pi m^2 p} + \frac{2h\sin(mp)}{\pi m^2(\pi - p)} \\ &= \frac{2h\sin(mp)}{\pi m^2}\left(\frac{1}{p} + \frac{1}{\pi - p}\right) \\ &= \frac{2h\sin(mp)}{\pi m^2}\frac{\pi}{p(m - p)} \\ &= \frac{2h}{m^2}\frac{\sin(mp)}{p(m - p)}, \end{aligned}$$

as desired. The second, fourth, ... harmonics are missing when we have $A_{2n} = 0$ for all positive integers n. The formula becomes

$$A_{2n} = \frac{2h}{4n^2}\frac{\sin(2np)}{p(\pi - p)}.$$

So the second, fourth, ... harmonics are missing whenever we have $\sin(2np) = 0$, or equivalently $p = \frac{\pi}{2}$. Similarly, the third, sixth, ... harmonics are missing when we have $A_{3n} = 0$ for all positive integers n. The formula becomes

$$A_{3n} = \frac{2h}{9n^2}\frac{\sin(3np)}{p(\pi - p)}.$$

So the second, fourth, ... harmonics are missing whenever we have $\sin(3np) = 0$, or equivalently $p = \frac{\pi}{3}$. □

1.3.10. Show that the expression of the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is given in polar coordinates by the formula

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Also, prove that

$$\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right| + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2.$$

Solution. We know already that the Laplacian is defined in the Cartesian coordinate system by

$$\begin{aligned}\Delta u &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.\end{aligned}$$

To compute the Laplace equation $\Delta u = 0$ in the polar coordinate system, we need to derive the equivalent expression of the Laplacian in polar coordinates. Let

$$\begin{aligned}x = x(r,\theta) &= r\cos(\theta), \\ y = y(r,\theta) &= r\sin(\theta), \\ u(x,y) = u(r,\theta) &= u(x(r,\theta), y(r,\theta)),\end{aligned}$$

the first two of which imply

$$\begin{aligned}r &= \sqrt{x^2+y^2}, \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right).\end{aligned}$$

We obtain first partial derivatives

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}(\sqrt{x^2+y^2}) = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}, \\ \frac{\partial r}{\partial y} &= \frac{\partial}{\partial y}(\sqrt{x^2+y^2}) = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}, \\ \frac{\partial \theta}{\partial x} &= \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = -\frac{y}{x^2+y^2} = -\frac{y}{r^2}, \\ \frac{\partial \theta}{\partial y} &= \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{x}{x^2+y^2} = \frac{x}{r^2}\end{aligned}$$

and the second partial derivatives

$$\begin{aligned}\frac{\partial^2 r}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{y^2}{r^3}, \\ \frac{\partial^2 r}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{x^2}{r^3}, \\ \frac{\partial^2 \theta}{\partial x^2} &= \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2} = \frac{2xy}{r^4}, \\ \frac{\partial^2 \theta}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = -\frac{2xy}{(x^2+y^2)^2} = -\frac{2xy}{r^4}.\end{aligned}$$

So, by the multivariable chain rule, we obtain the first partial derivatives

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}(u(r,\theta)) \\ &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= \frac{\partial u}{\partial r}\frac{x}{r} - \frac{\partial u}{\partial \theta}\frac{y}{r^2}\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}(u(r,\theta)) \\
&= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \\
&= \frac{\partial u}{\partial r}\frac{y}{r} + \frac{\partial u}{\partial \theta}\frac{x}{r^2}
\end{aligned}
$$

and the second partial derivatives

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2}{\partial x^2}(u(r,\theta)) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\frac{\partial}{\partial x}\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\right) \\
&= \left(\frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2}\right) + \left(\frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}\right) \\
&= \frac{\partial^2 u}{\partial r^2}\frac{x^2}{r^2} + \frac{\partial u}{\partial r}\frac{y^2}{r^3} + \frac{\partial^2 u}{\partial \theta^2}\frac{y^2}{r^4} + \frac{\partial u}{\partial \theta}\frac{2xy}{r^4}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2}{\partial y^2}(u(r,\theta)) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}\right) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}\right) + \frac{\partial}{\partial y}(\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}\Bigg) \\
&= \left(\frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2 + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial y^2}\right) + \left(\frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2}\right) \\
&= \frac{\partial^2 u}{\partial r^2}\frac{y^2}{r^2} + \frac{\partial u}{\partial r}\frac{x^2}{r^3} + \frac{\partial^2 u}{\partial \theta^2}\frac{x^2}{r^4} - \frac{\partial u}{\partial \theta}\frac{2xy}{r^4}.
\end{aligned}
$$

Therefore, the Laplacian in polar coordinates is

$$
\begin{aligned}
\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&= \left(\frac{\partial^2 u}{\partial r^2}\frac{x^2}{r^2} + \frac{\partial u}{\partial r}\frac{y^2}{r^3} + \frac{\partial^2 u}{\partial \theta^2}\frac{y^2}{r^4} + \frac{\partial u}{\partial \theta}\frac{2xy}{r^4}\right) + \left(\frac{\partial^2 u}{\partial r^2}\frac{y^2}{r^2} + \frac{\partial u}{\partial r}\frac{x^2}{r^3} + \frac{\partial^2 u}{\partial \theta^2}\frac{x^2}{r^4} - \frac{\partial u}{\partial \theta}\frac{2xy}{r^4}\right) \\
&= \frac{\partial^2 u}{\partial r^2}\frac{x^2+y^2}{r^2} + \frac{\partial u}{\partial r}\frac{x^2+y^2}{r^3} + \frac{\partial^2 u}{\partial \theta^2}\frac{x^2+y^2}{r^4} \\
&= \frac{\partial^2 u}{\partial r^2}\frac{r^2}{r^2} + \frac{\partial u}{\partial r}\frac{r^2}{r^3} + \frac{\partial^2 u}{\partial \theta^2}\frac{r^2}{r^4} \\
&= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial u}{\partial \theta} \\
&= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta}\right)u,
\end{aligned}
$$

and so the Laplacian in polar coordinates is given by

$$
\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta}.
$$

Also, we obtain

$$\begin{aligned}
\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 &= \left|\frac{\partial u}{\partial r}\frac{x}{r} - \frac{\partial u}{\partial \theta}\frac{y}{r^2}\right|^2 + \left|\frac{\partial u}{\partial r}\frac{y}{r} + \frac{\partial u}{\partial \theta}\frac{x}{r^2}\right|^2 \\
&= \left(\left|\frac{\partial u}{\partial r}\right|^2\frac{x^2}{r^2} - 2\left|\frac{\partial u}{\partial r}\right|\frac{xy}{r^3} + \left|\frac{\partial u}{\partial \theta}\right|^2\frac{y^2}{r^4}\right) + \left(\left|\frac{\partial u}{\partial r}\right|^2\frac{y^2}{r^2} + 2\left|\frac{\partial u}{\partial r}\right|\frac{xy}{r^3} + \left|\frac{\partial u}{\partial \theta}\right|^2\frac{x^2}{r^4}\right) \\
&= \left|\frac{\partial u}{\partial r}\right|^2\frac{x^2+y^2}{r^2} + \left|\frac{\partial u}{\partial \theta}\right|^2\frac{x^2+y^2}{r^4} \\
&= \left|\frac{\partial u}{\partial r}\right|^2\frac{r^2}{r^2} + \left|\frac{\partial u}{\partial \theta}\right|^2\frac{r^2}{r^4} \\
&= \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2,
\end{aligned}$$

as desired. □

1.4.1. We look for a solution of the steady-state heat equation $\Delta u = 0$ in the rectangle $R = \{(x, y) : 0 \le x \le \pi, 0 \le y \le 1\}$ that vanishes on the vertical sides of R, and so that

$$u(x, 0) = f_0(x) \text{ and } u(x, 1) = f_1(x),$$

where f_0 and f_1 are initial data which fix the temperature distribution on the horizontal sides of the rectangle. Use separation of variables to show that if f_0 and f_1 have Fourier expansions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin(kx) \text{ and } f_1(x) = \sum_{k=1}^{\infty} B_k \sin(kx)$$

then

$$u(x, y) = \sum_{k=1}^{\infty}\left(\frac{\sinh(k(1-y))}{\sinh(k)}A_k + \frac{\sinh(ky)}{\sinh(k)}B_k\right)\sinh(kx).$$

We recall the definitions of the hyperbolic sine and cosine functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \text{ and } \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Solution. See my Homework 2 solutions. This problem is required in Homework 2. So I wrote a solution for it. □

2.6.1. Suppose f is 2π-periodic and integrable on any finite interval. Prove that if $a, b \in \mathbb{R}$, then

$$\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx = \int_{a-2\pi}^{b-2\pi} f(x)\,dx.$$

Also prove that

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx.$$

Solution. Since f is 2π-periodic, we have $f(x-2\pi) = f(x) = f(x+2\pi)$. We will employ the substitution rule from first-year calculus. If we let $u = x - 2\pi$, which implies $du = dx$ and $x = u + 2\pi$, then we obtain

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_{a+2\pi}^{b+2\pi} f(u+2\pi)\,du \\
&= \int_{a+2\pi}^{b+2\pi} f(x+2\pi)\,dx \\
&= \int_{a+2\pi}^{b+2\pi} f(x)\,dx.
\end{aligned}$$

Similarly, if we let $u = x + 2\pi$, which implies $du = dx$ and $x = u - 2\pi$, then we obtain

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_{a-2\pi}^{b-2\pi} f(u-2\pi)\,du \\
&= \int_{a-2\pi}^{b-2\pi} f(x-2\pi)\,dx \\
&= \int_{a+2\pi}^{b+2\pi} f(x)\,dx.
\end{aligned}$$

Now, using these integral inequalities that we proved, if we let $u = x + a$, which implies $du = dx$ and $x = u - a$, then we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} f(x+a)\,dx &= \int_{-\pi+a}^{\pi+a} f(u)\,du \\ &= \int_{-\pi+a}^{\pi+a} f(x)\,dx.\end{aligned}$$

In particular, from the first set of equalities we have

$$\begin{aligned}\int_{\pi}^{\pi+a} f(x)\,dx &= \int_{\pi-2\pi}^{\pi+a-2\pi} f(x)\,dx \\ &= \int_{-\pi}^{-\pi+a} f(x)\,dx,\end{aligned}$$

which implies

$$\begin{aligned}\int_{-\pi+a}^{\pi+a} f(x)\,dx &= \int_{-\pi}^{\pi} f(x)\,dx + \int_{\pi}^{\pi+a} f(x)\,dx - \int_{-\pi}^{-\pi+a} f(x)\,dx \\ &= \int_{-\pi}^{\pi} f(x)\,dx + \int_{-\pi}^{-\pi+a} f(x)\,dx - \int_{-\pi}^{-\pi+a} f(x)\,dx \\ &= \int_{-\pi}^{\pi} f(x)\,dx,\end{aligned}$$

as desired. □

2.6.2. In this exercise we show how the symmetries of a function imply certain properties of its Fourier coefficients. Let f be a 2π-periodic Riemann integrable function defined on $\mathbb{R}$.

(a) Show that the Fourier series of the function f can be written as

$$f(\theta) \sim \hat{f}(0) + \sum_{n=1}^{\infty}((\hat{f}(n) + \hat{f}(-n))\cos(n\theta) + i(\hat{f}(n) - \hat{f}(-n))\sin(n\theta)).$$

Solution. According to page 34 of the textbook, the Fourier series of $f : \mathbb{R} \to \mathbb{R}$ (in exponential form) is given formally by

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{\frac{2\pi i n\theta}{L}}$$

where we define $L := b - a$ and

$$\hat{f}(n) := \frac{1}{L}\int_a^b f(\theta)e^{-\frac{2\pi i n\theta}{L}}\,d\theta.$$

Observe that, given any $\hat{f}(n)$, the expression

$$h_{\text{even}}(n) := \frac{\hat{f}(n) + \hat{f}(-n)}{2}$$

is an even function of n because $h_{\text{even}}(n)$ satisfies

$$\begin{aligned}h_{\text{even}}(-n) &= \frac{\hat{f}(-n) + \hat{f}(-(-n))}{2} \\ &= \frac{\hat{f}(-n) + \hat{f}(n)}{2} \\ &= \frac{\hat{f}(n) + \hat{f}(-n)}{2} \\ &= h_{\text{even}}(n),\end{aligned}$$

and the expression

$$h_{\text{odd}}(n) := \frac{\hat{f}(n) - \hat{f}(-n)}{2}$$

is odd in n because $h_{\text{odd}}(n)$ satisfies

$$\begin{aligned}h_{\text{odd}}(-n) &= \frac{\hat{f}(-n) - \hat{f}(-(-n))}{2} \\ &= \frac{\hat{f}(-n) - \hat{f}(n)}{2} \\ &= -\frac{\hat{f}(n) - \hat{f}(-n)}{2} \\ &= -h_{\text{odd}}(n).\end{aligned}$$

Furthermore, we can write $\hat{f}(n)$ and $\hat{f}(-n)$ as a decomposition of even and odd functions

$$\begin{aligned}\hat{f}(n) &= \frac{\hat{f}(n)+\hat{f}(-n)}{2} + \frac{\hat{f}(n)-\hat{f}(-n)}{2} \\ &= h_{\text{even}}(n) + h_{\text{odd}}(n).\end{aligned}$$

and

$$\begin{aligned}\hat{f}(-n) &= \frac{\hat{f}(n)+\hat{f}(-n)}{2} - \frac{\hat{f}(n)-\hat{f}(-n)}{2} \\ &= h_{\text{even}}(n) - h_{\text{odd}}(n).\end{aligned}$$

Using the formal definition of the Fourier series for $L := 2\pi$ and Euler's formula, we have

$$\begin{aligned}
f(\theta) &\sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} \\
&= \hat{f}(0)e^{i(0)\theta} + \sum_{n=1}^{\infty} \hat{f}(n)e^{in\theta} + \sum_{n=-\infty}^{-1} \hat{f}(n)e^{in\theta} \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} \hat{f}(n)e^{in\theta} + \sum_{n=1}^{\infty} \hat{f}(-n)e^{i(-n)\theta} \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n)e^{in\theta} + \hat{f}(-n)e^{-in\theta}) \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} (h_{\text{even}}(n) + h_{\text{odd}}(n))e^{in\theta} + (h_{\text{even}}(n) - h_{\text{odd}}(n))e^{-in\theta}) \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} (h_{\text{even}}(n)(e^{in\theta} + e^{-in\theta}) + h_{\text{odd}}(n)(e^{in\theta} - e^{-in\theta})) \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} h_{\text{even}}(n)(e^{in\theta} + e^{-in\theta}) + \sum_{n=1}^{\infty} h_{\text{odd}}(n)(e^{in\theta} - e^{-in\theta}) \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} 2h_{\text{even}}(n)\frac{e^{in\theta} + e^{-in\theta}}{2} + i\sum_{n=1}^{\infty} 2h_{\text{odd}}(n)\frac{e^{in\theta} - e^{-in\theta}}{2i} \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} 2h_{\text{even}}(n)\cos(n\theta) + i\sum_{n=1}^{\infty} 2h_{\text{odd}}(n)\sin(n\theta) \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} 2\frac{\hat{f}(n)+\hat{f}(-n)}{2}\cos(n\theta) + i\sum_{n=1}^{\infty} 2\frac{\hat{f}(n)-\hat{f}(-n)}{2}\sin(n\theta) \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n)+\hat{f}(-n))\cos(n\theta) + i\sum_{n=1}^{\infty} (\hat{f}(n)-\hat{f}(-n))\sin(n\theta),
\end{aligned}$$

as desired. □

(b) Prove that if f is even, then $\hat{f}(n) = \hat{f}(-n)$, and we get a cosine series.

Solution. Since f is even, we have $f(-\theta) = f(\theta)$ for all $\theta \in \mathbb{R}$. Also recall that $\cos(n\theta)$ is an even function of n and $\sin(n\theta)$ is odd in n for all $n \in \mathbb{R}$, meaning we have $\cos(-n) = \cos(n)$ and $\sin(-n) = -\sin(n)$, respectively. Using the formula of $\hat{f}(n)$ from part (a), we obtain

$$\begin{aligned}
f(-\theta) &\sim \hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n)+\hat{f}(-n))\cos(-n\theta) + i\sum_{n=1}^{\infty} (\hat{f}(n)-\hat{f}(-n))\sin(-n\theta) \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n)+\hat{f}(-n))\cos(n\theta) + i\sum_{n=1}^{\infty} (\hat{f}(n)-\hat{f}(-n))(-\sin(n\theta)) \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n)+\hat{f}(-n))\cos(n\theta) + i\sum_{n=1}^{\infty} (\hat{f}(-n)-\hat{f}(n))\sin(n\theta).
\end{aligned}$$

From this and the expression of $f(\theta)$, we obtain

$$\begin{aligned}
&\hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n) + \hat{f}(-n)) \cos(n\theta) + i \sum_{n=1}^{\infty} (\hat{f}(-n) - \hat{f}(n)) \sin(n\theta) \\
&\sim f(-\theta) \\
&= f(\theta) \\
&\sim \hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n) + \hat{f}(-n)) \cos(n\theta) + i \sum_{n=1}^{\infty} (\hat{f}(n) - \hat{f}(-n)) \sin(n\theta)
\end{aligned}$$

which algebraically simplifies to

$$\sum_{n=1}^{\infty} (\hat{f}(n) - \hat{f}(-n)) \sin(n\theta) \sim 0,$$

We know from linear algebra that the basis of smooth functions $\{\sin(n\theta)\}_{n=1}^{\infty} \subset C^{\infty}(\mathbb{R})$ is a linearly independent set, and so we must conclude $\hat{f}(n) - \hat{f}(-n) = 0$, or $\hat{f}(-n) = \hat{f}(n)$, signifying that $\hat{f}(n)$ is an even function of n. □

(c) Prove that if f is odd, then $\hat{f}(n) = -\hat{f}(-n)$, and we get a sine series.

Solution. Since f is odd, we have $f(-\theta) = -f(\theta)$ for all $\theta \in \mathbb{R}$. Also recall that $\cos(n\theta)$ is an even function of n and $\sin(n\theta)$ is odd in n for all $n \in \mathbb{R}$, meaning we have $\cos(-n) = \cos(n)$ and $\sin(-n) = -\sin(n)$, respectively. Using the formula of $\hat{f}(n)$ from part (a), we obtain

$$\begin{aligned}
f(-\theta) &\sim \hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n) + \hat{f}(-n)) \cos(-n\theta) + i \sum_{n=1}^{\infty} (\hat{f}(n) - \hat{f}(-n)) \sin(-n\theta) \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n) + \hat{f}(-n)) \cos(n\theta) + i \sum_{n=1}^{\infty} (\hat{f}(n) - \hat{f}(-n))(-\sin(n\theta)) \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n) + \hat{f}(-n)) \cos(n\theta) + i \sum_{n=1}^{\infty} (\hat{f}(-n) - \hat{f}(n))(\sin(n\theta)).
\end{aligned}$$

From this and the expression of $f(\theta)$, we obtain

$$\begin{aligned}
&\hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n) + \hat{f}(-n)) \cos(n\theta) + i \sum_{n=1}^{\infty} (\hat{f}(-n) - \hat{f}(n)) \sin(n\theta) \\
&= f(-\theta) \\
&= -f(\theta) \\
&\sim -\left(\hat{f}(0) + \sum_{n=1}^{\infty} (\hat{f}(n) + \hat{f}(-n)) \cos(n\theta) + i \sum_{n=1}^{\infty} (\hat{f}(n) - \hat{f}(-n)) \sin(n\theta)\right) \\
&= -\hat{f}(0) - \sum_{n=1}^{\infty} (\hat{f}(n) + \hat{f}(-n)) \cos(n\theta) + i \sum_{n=1}^{\infty} (\hat{f}(-n) - \hat{f}(n)) \sin(n\theta)
\end{aligned}$$

which algebraically simplifies to

$$\sum_{n=1}^{\infty} (\hat{f}(n) + \hat{f}(-n)) \cos(n\theta) \sim 0,$$

We know from linear algebra that the basis of smooth functions $\{\cos(n\theta)\}_{n=1}^{\infty} \subset C^{\infty}(\mathbb{R})$ is a linearly independent set, and so we must conclude $\hat{f}(n) + \hat{f}(-n) = 0$, or $\hat{f}(-n) = -\hat{f}(n)$, signifying that $\hat{f}(n)$ is an odd function of n. □

(d) Suppose that $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbb{R}$. Show that $\hat{f}(n) = 0$ for all odd n.

Solution. Since $f(\theta)$ is 2π-periodic, we have $L := 2\pi$, and Exercise 2.6.1 implies that the Fourier coefficient is

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{L} \int_{a}^{a+L} f(\theta) e^{-\frac{2\pi i n\theta}{L}} \, d\theta \\
&= \frac{1}{2\pi} \int_{a}^{a+2\pi} f(\theta) e^{-\frac{2\pi i n\theta}{2\pi}} \, d\theta \\
&= \frac{1}{2\pi} \int_{a}^{a+2\pi} f(\theta) e^{-in\theta} \, d\theta
\end{aligned}$$

for any $a \in \mathbb{R}$. Also, for all odd n, which allows us to write $n = 2k + 1$ for any integer k, we have

$$\begin{aligned} e^{in\pi} &= e^{i(2k+1)\pi} \\ &= e^{2\pi ik+\pi i} \\ &= e^{2\pi ik} e^{\pi i} \\ &= (e^{2\pi i})^k e^{\pi i} \\ &= 1^k \cdot (-1) \\ &= -1. \end{aligned}$$

Using the given assumption $f(\theta + \pi) = f(\theta)$ and our employed substitution $u = \theta + \pi$, we have

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \int_{a-\pi}^{a+\pi} f(\theta) e^{-in\theta} \, d\theta \\ &= \frac{1}{2\pi} \int_{a-\pi}^{a} f(\theta) e^{-in\theta} \, d\theta + \frac{1}{2\pi} \int_{a}^{a+\pi} f(\theta) e^{-in\theta} \, d\theta \\ &= \frac{1}{2\pi} \int_{a-\pi}^{a} f(\theta+\pi) e^{-in\theta} \, d\theta + \frac{1}{2\pi} \int_{a}^{a+\pi} f(\theta) e^{-in\theta} \, d\theta \\ &= \frac{1}{2\pi} \int_{a}^{a+\pi} f(u) e^{-in(u-\pi)} \, du + \frac{1}{2\pi} \int_{a}^{a+\pi} f(\theta) e^{-in\theta} \, d\theta \\ &= \frac{1}{2\pi} \int_{a}^{a+\pi} f(\theta) e^{-in(\theta-\pi)} \, d\theta + \frac{1}{2\pi} \int_{a}^{a+\pi} f(\theta) e^{-in\theta} \, d\theta \\ &= \frac{1}{2\pi} \int_{a}^{a+\pi} f(\theta) e^{-in\theta} e^{in\pi} \, d\theta + \frac{1}{2\pi} \int_{a}^{a+\pi} f(\theta) e^{-in\theta} \, d\theta \\ &= \frac{1}{2\pi} \int_{a}^{a+\pi} f(\theta) e^{-in\theta} (e^{in\pi} + 1) \, d\theta \\ &= \frac{1}{2\pi} \int_{a}^{a+\pi} f(\theta) e^{-in\theta} (-1 + 1) \, d\theta \\ &= \frac{1}{2\pi} \int_{a}^{a+\pi} f(\theta) e^{-in\theta} (0) \, d\theta \\ &= 0, \end{aligned}$$

as desired. □

(e) Show that f is real-valued if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all integers n.

Solution. Suppose f is real-valued; that is, assume $\overline{f(\theta)} = f(\theta)$, where $\overline{f(\theta)}$ denotes the complex conjugate of $f(\theta)$. Applying the usual properties of complex conjugation, we have

$$\begin{aligned} \overline{\hat{f}(n)} &= \overline{\frac{1}{\pi} \int_{a}^{a+2\pi} f(\theta) e^{-in\theta} \, d\theta} \\ &= \frac{1}{\pi} \int_{a}^{a+2\pi} \overline{f(\theta) e^{-in\theta}} \, \overline{d\theta} \\ &= \frac{1}{\pi} \int_{a}^{a+2\pi} \overline{f(\theta)} \, \overline{e^{-in\theta}} \, d\theta \\ &= \frac{1}{\pi} \int_{a}^{a+2\pi} f(\theta) e^{in\theta} \, d\theta \\ &= \frac{1}{\pi} \int_{a}^{a+2\pi} f(\theta) e^{-i(-n)\theta} \, d\theta \\ &= \hat{f}(-n) \end{aligned}$$

for all integers n.

Conversely, assume $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all integers n. Applying the usual properties of complex conjugation and the formal definition of a Fourier series, and employing the substitution $m := -n$, we have

$$\begin{aligned}
\overline{f(\theta)} &\sim \overline{\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}} \\
&= \sum_{n=-\infty}^{\infty} \overline{\hat{f}(n)e^{in\theta}} \\
&= \sum_{n=-\infty}^{\infty} \overline{\hat{f}(n)}\,\overline{e^{in\theta}} \\
&= \sum_{n=-\infty}^{\infty} \hat{f}(-n)e^{-in\theta} \\
&= \sum_{m=-\infty}^{\infty} \hat{f}(m)e^{im\theta} \\
&\sim f(\theta),
\end{aligned}$$

meaning that $\overline{f(\theta)}$ and $f(\theta)$ are equal up to a scaling factor with a Fourier coefficient. But this is enough to conclude that f is real-valued. □

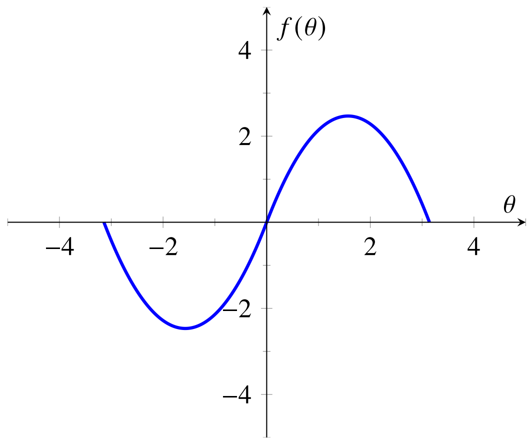
2.6.4. Consider the 2π-periodic odd function defined on $[0, \pi]$ by $f(\theta) = \theta(\pi - \theta)$.

(a) Draw the graph of f.

Solution. Since we are given that the 2π-periodic odd function is only defined on $[0, \pi]$, we can employ an odd extension to obtain the resulting 2π-periodic odd function defined on $[-\pi, \pi]$:

$$f(\theta) = \begin{cases} \theta(\pi - \theta) & \text{if } 0 \le \theta \le \pi, \\ \theta(\pi + \theta) & \text{if } -\pi \le \theta \le 0. \end{cases}$$

This is enough to graph one complete cycle of the 2π-periodic odd function on $[-\pi, \pi]$.



□

(b) Compute the Fourier coefficients of f, and show that

$$f(\theta) = \frac{8}{\pi} \sum_{k=1,3,5,\ldots} \frac{\sin(k\theta)}{k^3}.$$

Solution. If $n \neq 0$, then we can apply the method of integration by parts to obtain

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{0} f(\theta)e^{-\frac{2\pi i n\theta}{2\pi}}\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{0}\theta(\pi+\theta)e^{-in\theta}\,d\theta \\
&= \frac{1}{2\pi in}\int_{-\pi}^{0}\theta(\pi+\theta)d(e^{-in\theta}) \\
&= -\frac{1}{2\pi in}\left(\theta(\pi+\theta)e^{-in\theta}\big|_{-\pi}^{0} - \int_{-\pi}^{0} e^{-in\theta}d(\theta(\pi+\theta))\right) \\
&= -\frac{1}{2\pi in}\left((0-0) - \int_{-\pi}^{0} e^{-in\theta}((\pi+\theta)\,d\theta + \theta\,d(\pi+\theta))\right) \\
&= \frac{1}{2\pi in}\int_{-\pi}^{0} e^{-in\theta}((\pi+\theta)\,d\theta + \theta\,d\theta)) \\
&= \frac{1}{2\pi in}\int_{-\pi}^{0} e^{-in\theta}(\pi+2\theta)\,d\theta \\
&= -\frac{1}{2\pi i^2 n^2}\int_{-\pi}^{0} \pi+2\theta\,d(e^{-in\theta}) \\
&= \frac{1}{2\pi n^2}\left((\pi+2\theta)e^{-in\theta}\big|_{-\pi}^{0} - \int_{-\pi}^{0} e^{-in\theta}d(\pi+2\theta)\right) \\
&= \frac{1}{2\pi n^2}\left((\pi-(-\pi)(-1)^n) - 2\int_{-\pi}^{0} e^{-in\theta}\,d\theta\right) \\
&= \frac{1}{2\pi n^2}\left(\pi(1+(-1)^n) - \frac{2}{-in}e^{-in\theta}\bigg|_{-\pi}^{0}\right) \\
&= \frac{1}{2\pi n^2}\left(\pi(1+(-1)^n) - \frac{2i}{n}(1-(-1)^n)\right) \\
&= \frac{1}{2\pi n^2}\left(\pi(1+(-1)^n) + \frac{2i}{n}((-1)^n-1)\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2\pi}\int_{0}^{\pi} f(\theta)e^{-\frac{2\pi i n\theta}{2\pi}}\,d\theta &= \frac{1}{2\pi}\int_{0}^{\pi}\theta(\pi-\theta)e^{-in\theta}\,d\theta \\
&= \frac{1}{2\pi in}\int_{0}^{\pi}\theta(\pi-\theta)d(e^{-in\theta}) \\
&= -\frac{1}{2\pi in}\left(\theta(\pi-\theta)e^{-in\theta}\big|_{0}^{\pi} - \int_{0}^{\pi} e^{-in\theta}d(\theta(\pi-\theta))\right) \\
&= -\frac{1}{2\pi in}\left((0-0) - \int_{0}^{\pi} e^{-in\theta}((\pi-\theta)\,d\theta + \theta\,d(\pi-\theta))\right) \\
&= \frac{1}{2\pi in}\int_{0}^{\pi} e^{-in\theta}((\pi-\theta)\,d\theta - \theta\,d\theta)) \\
&= \frac{1}{2\pi in}\int_{0}^{\pi} e^{-in\theta}(\pi-2\theta)\,d\theta \\
&= -\frac{1}{2\pi i^2 n^2}\int_{0}^{\pi} \pi-2\theta\,d(e^{-in\theta}) \\
&= \frac{1}{2\pi n^2}\left((\pi-2\theta)e^{-in\theta}\big|_{0}^{\pi} - \int_{0}^{\pi} e^{-in\theta}d(\pi-2\theta)\right) \\
&= \frac{1}{2\pi n^2}\left((-\pi(-1)^n-\pi) + 2\int_{0}^{\pi} e^{-in\theta}\,d\theta\right) \\
&= \frac{1}{2\pi n^2}\left(-\pi((-1)^n+1) + \frac{2}{-in}e^{-in\theta}\bigg|_{0}^{\pi}\right) \\
&= \frac{1}{2\pi n^2}\left(-\pi((-1)^n+1) + \frac{2i}{n}((-1)^n-1)\right),
\end{aligned}$$

which imply

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-\frac{2\pi in\theta}{2\pi}}\,d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} f(\theta)e^{-\frac{2\pi in\theta}{2\pi}}\,d\theta + \frac{1}{2\pi}\int_{0}^{\pi} f(\theta)e^{-\frac{2\pi in\theta}{2\pi}}\,d\theta \\
&= \frac{1}{2\pi n^2}\left(\pi(1+(-1)^n) + \frac{2i}{n}((-1)^n-1)\right) + \frac{1}{2\pi n^2}\left(-\pi((-1)^n+1) + \frac{2i}{n}((-1)^n-1)\right) \\
&= \frac{4i}{2\pi n^3}((-1)^n-1) \\
&= \frac{2((-1)^n-1)}{\pi n^3}i
\end{aligned}
$$

If $n = 0$, then we obtain

$$
\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-\frac{2\pi i(0)\theta}{2\pi}}\,d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} \theta(\pi+\theta)\,d\theta + \frac{1}{2\pi}\int_{0}^{\pi} \theta(\pi-\theta)\,d\theta \\
&= \frac{1}{2\pi}\left(\pi\int_{-\pi}^{0}\theta\,d\theta + \int_{-\pi}^{0}\theta^2\,d\theta\right) + \frac{1}{2\pi}\left(\pi\int_{0}^{\pi}\theta\,d\theta - \int_{0}^{\pi}\theta^2\,d\theta\right) \\
&= \frac{1}{2\pi}\left(\frac{\pi}{2}\theta^2\Big|_{-\pi}^{0} + \frac{1}{3}\theta^3\Big|_{-\pi}^{0}\right) + \frac{1}{2\pi}\left(\frac{\pi}{2}\theta^2\Big|_{0}^{\pi} - \frac{1}{3}\theta^3\Big|_{0}^{\pi}\right) \\
&= \frac{1}{2\pi}\left(-\frac{\pi^3}{2} + \frac{\pi^3}{3}\right) + \frac{1}{2\pi}\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) \\
&= 0.
\end{aligned}
$$

So the Fourier series is

$$
\begin{aligned}
f(\theta) &= \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi in\theta}{2\pi}} \\
&= c_0 e^{-i(0)\theta} + \sum_{n\neq 0} c_n e^{in\theta} \\
&= 0 + \sum_{n\neq 0} \frac{2((-1)^n-1)}{\pi n^3} i e^{in\theta} \\
&= \sum_{n\neq 0} \frac{2((-1)^n-1)}{\pi n^3} i e^{in\theta} \\
&= \sum_{n=-\infty}^{-1} \frac{2((-1)^n-1)}{\pi n^3} i e^{in\theta} + \sum_{n=1}^{\infty} \frac{2((-1)^n-1)}{\pi n^3} i e^{in\theta} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{-n}-1}{\pi(-n)^3} i e^{i(-n)\theta} + \sum_{n=1}^{\infty} \frac{2((-1)^n-1)}{\pi n^3} i e^{in\theta} \\
&= \sum_{n=1}^{\infty} -\frac{2((-1)^n-1)}{\pi n^3} i e^{-in\theta} + \sum_{n=1}^{\infty} \frac{2((-1)^n-1)}{\pi n^3} i e^{in\theta} \\
&= \sum_{n=1}^{\infty} \frac{2((-1)^n-1)}{\pi n^3} i(e^{in\theta} - e^{-in\theta}) \\
&= \sum_{n=1}^{\infty} -2\frac{2((-1)^n-1)}{\pi n^3}\frac{e^{in\theta}-e^{-in\theta}}{2i} \\
&= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3}\sin(n\theta) \\
&= \frac{4}{\pi}\sum_{n=1,3,5,\ldots} \frac{2}{n^3}\sin(n\theta) \\
&= \frac{8}{\pi}\sum_{n=1,3,5,\ldots} \frac{\sin(n\theta)}{n^3},
\end{aligned}
$$

as desired. □

2.6.5. On the interval $[-\pi, \pi]$, consider the function

$$f(\theta) = \begin{cases} 0 & \text{if } |\theta| > \delta, \\ 1 - \frac{|\theta|}{\delta} & \text{if } |\theta| \le \delta. \end{cases}$$

Thus the graph of f has the shape of a triangular tent. Show that

$$f(\theta) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1-\cos(n\delta)}{n^2\pi\delta}\cos(n\theta).$$

Solution. If $n \neq 0$, then we have

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-\frac{2\pi i n\theta}{2\pi}}\,d\theta \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} 0e^{-in\theta}\,d\theta + \int_{-\delta}^{\delta}\left(1-\frac{|\theta|}{\delta}\right)e^{-in\theta}\,d\theta + \int_{\delta}^{\pi} 0e^{-in\theta}\,d\theta\right) \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta}\left(1-\frac{|\theta|}{\delta}\right)e^{-in\theta}\,d\theta \\
&= \frac{1}{2\pi}\left(\int_{-\delta}^{0}\left(1+\frac{\theta}{\delta}\right)e^{-in\theta}\,d\theta + \int_{0}^{\delta}\left(1-\frac{\theta}{\delta}\right)e^{-in\theta}\,d\theta\right) \\
&= \frac{1}{2\pi}\left(\left(-\frac{1}{in}\left(1+\frac{\theta}{\delta}\right)e^{-in\theta}\Big|_{-\delta}^{0} + \frac{1}{in}\int_{-\delta}^{0} e^{-in\theta}\,d\theta\right) + \left(-\frac{1}{in}\left(1-\frac{\theta}{\delta}\right)e^{-in\theta}\Big|_{0}^{\delta} - \frac{1}{in}\int_{0}^{\delta} e^{-in\theta}\,d\theta\right)\right) \\
&= \frac{1}{2\pi}\left(\left(-\frac{1}{in} + \frac{1}{in\delta}\int_{-\delta}^{0} e^{-in\theta}\,d\theta\right) + \left(\frac{1}{in} - \frac{1}{in\delta}\int_{0}^{\delta} e^{-in\theta}\,d\theta\right)\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{in\delta}\int_{-\delta}^{0} e^{-in\theta}\,d\theta - \frac{1}{in\delta}\int_{0}^{\delta} e^{-in\theta}\,d\theta\right) \\
&= \frac{1}{2\pi}\left(-\frac{1}{i^2n^2\delta}e^{-in\theta}\Big|_{-\delta}^{0} - \frac{1}{i^2n^2\delta}e^{-in\theta}\Big|_{0}^{\delta}\right) \\
&= \frac{1}{2\pi n^2\delta}((1-e^{in\delta}) - (e^{-in\delta}-1)) \\
&= \frac{2-e^{in\delta}-e^{-in\delta}}{2\pi n^2\delta} \\
&= \frac{2}{2\pi n^2\delta} - \frac{1}{\pi n^2\delta}\frac{e^{in\delta}+e^{-in\delta}}{2} \\
&= \frac{1}{\pi n^2\delta} - \frac{\cos(n\delta)}{\pi n^2\delta} \\
&= \frac{1-\cos(n\delta)}{\pi n^2\delta}.
\end{aligned}$$

If $n = 0$, then we have

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-\frac{2\pi i(0)\theta}{2\pi}}\,d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} 0\,d\theta + \int_{-\delta}^{\delta} 1-\frac{|\theta|}{\delta}\,d\theta + \int_{\delta}^{\pi} 0\,d\theta\right) \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} 1-\frac{|\theta|}{\delta}\,d\theta \\
&= \frac{1}{2\pi}\left(\frac{1}{2}(2\delta)(1)\right) \\
&= \frac{\delta}{2\pi}.
\end{aligned}$$

So the Fourier series is

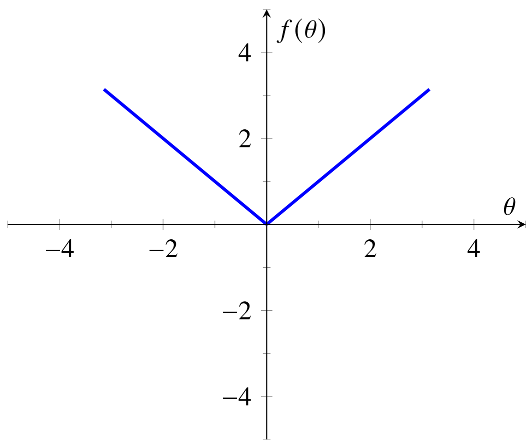
$$\begin{aligned}
f(\theta) &= \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n\theta}{2\pi}} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \\
&= c_0 e^{i(0)\theta} + \sum_{n\neq 0} c_n e^{in\theta} \\
&= \frac{\delta}{2\pi} + \sum_{n\neq 0} \frac{1-\cos(n\delta)}{\pi n^2\delta} e^{in\theta} \\
&= \frac{\delta}{2\pi} + \sum_{n=-\infty}^{-1} \frac{1-\cos(n\delta)}{\pi n^2\delta} e^{in\theta} + \sum_{n=1}^{\infty} \frac{1-\cos(n\delta)}{\pi n^2\delta} e^{in\theta} \\
&= \frac{\delta}{2\pi} + \sum_{n=1}^{\infty} \frac{1-\cos(-n\delta)}{\pi(-n)^2\delta} e^{i(-n)\theta} + \sum_{n=1}^{\infty} \frac{1-\cos(n\delta)}{\pi n^2\delta} e^{in\theta} \\
&= \frac{\delta}{2\pi} + \sum_{n=1}^{\infty} \frac{1-\cos(n\delta)}{\pi n^2\delta} e^{-in\theta} + \sum_{n=1}^{\infty} \frac{1-\cos(n\delta)}{\pi n^2\delta} e^{in\theta} \\
&= \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1-\cos(n\delta)}{\pi n^2\delta} \frac{e^{in\theta} - e^{-in\theta}}{2} \\
&= \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1-\cos(n\delta)}{\pi n^2\delta} \cos(n\theta),
\end{aligned}$$

as desired. □

2.6.6. Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$.

(a) Draw the graph of f.

Solution.



□

(b) Calculate the Fourier coefficients of f, and show that

$$\hat{f}(n) = \begin{cases} \frac{\pi}{2} & \text{if } n = 0, \\ \frac{-1+(-1)^n}{\pi n^2} & \text{if } n \neq 0. \end{cases}$$

Solution. If $n \neq 0$, then we have

$$
\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-\frac{2\pi i n \theta}{2\pi}} \, d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\theta| e^{-in\theta} \, d\theta \\
&= \frac{1}{2\pi} \left(\int_{-\pi}^{0} -\theta e^{-in\theta} \, d\theta + \int_{0}^{\pi} \theta e^{-in\theta} \, d\theta \right) \\
&= \frac{1}{2\pi} \left(\left(\frac{1}{in} \theta e^{-in\theta} \Big|_{-\pi}^{0} - \frac{1}{in} \int_{-\pi}^{0} e^{-in\theta} \, d\theta \right) + \left(-\frac{1}{in} \theta e^{-in\theta} \Big|_{0}^{\pi} + \frac{1}{in} \int_{0}^{\pi} e^{-in\theta} \, d\theta \right) \right) \\
&= \frac{1}{2\pi} \left(\left(\frac{1}{in} (0 + \pi(-1)^n) + \frac{1}{i^2 n^2} e^{-in\theta} \Big|_{-\pi}^{0} \right) + \left(-\frac{1}{in} (\pi(-1)^n - 0) - \frac{1}{i^2 n^2} e^{-in\theta} \Big|_{0}^{\pi} \right) \right) \\
&= \frac{1}{2\pi} \left(-\frac{1}{n^2} e^{-in\theta} \Big|_{-\pi}^{0} + \frac{1}{n^2} e^{-in\theta} \Big|_{0}^{\pi} \right) \\
&= \frac{1}{2\pi n^2} (-(1 - (-1)^n) + ((-1)^n - 1)) \\
&= \frac{(-1)^n - 1}{\pi n^2}.
\end{aligned}
$$

If $n = 0$, then we have

$$
\begin{aligned}
c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-\frac{2\pi i (0) \theta}{2\pi}} \, d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\theta| \, d\theta \\
&= \frac{1}{2\pi} \left(\int_{-\pi}^{0} -\theta \, d\theta + \int_{0}^{\pi} \theta \, d\theta \right) \\
&= \frac{1}{2\pi} \left(\frac{\pi^2}{2} + \frac{\pi^2}{2} \right) \\
&= \frac{\pi}{2}.
\end{aligned}
$$

So the Fourier coefficient is

$$
\hat{f}(n) = c_n = \begin{cases} \frac{-1+(-1)^n}{\pi n^2} & \text{if } n \neq 0, \\ \frac{\pi}{2} & \text{if } n = 0 \end{cases}
$$

for all integers n. □

(c) What is the Fourier series of f in terms of sines and cosines?

Solution. The exponential form of the Fourier series is

$$
\begin{aligned}
f(\theta) &= \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n \theta}{2\pi}} \\
&= c_0 + \sum_{n \neq 0} c_n e^{in\theta} \\
&= \frac{\pi}{2} + \sum_{n \neq 0} \frac{-1 + (-1)^n}{\pi n^2} e^{in\theta}.
\end{aligned}
$$

Using the exponential form, we obtain

$$\begin{aligned}
f(\theta) &= \frac{\pi}{2} + \sum_{n\neq 0} \frac{-1+(-1)^n}{\pi n^2} e^{in\theta} \\
&= \frac{\pi}{2} + \sum_{n=-\infty}^{-1} \frac{-1+(-1)^n}{\pi n^2} e^{in\theta} + \sum_{n=1}^{\infty} \frac{-1+(-1)^n}{\pi n^2} e^{in\theta} \\
&= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-1+(-1)^{-n}}{\pi (-n)^2} e^{i(-n)\theta} + \sum_{n=1}^{\infty} \frac{-1+(-1)^n}{\pi n^2} e^{in\theta} \\
&= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-1+(-1)^n}{n^2} \frac{e^{in\theta}+e^{-in\theta}}{2} \\
&= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-1+(-1)^n}{n^2} \cos(\theta) \\
&= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos(\theta),
\end{aligned}$$

which is the Fourier series in its sine-cosine form. □

(d) Taking $\theta = 0$, prove that

$$\sum_{n=1,3,5,\ldots} \frac{1}{n^2} = \frac{\pi^2}{8} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution. At $\theta = 0$, we obtain

$$\begin{aligned}
0 &= |0| \\
&= f(0) \\
&= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos(0) \\
&= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n^2},
\end{aligned}$$

which is equivalent to

$$\sum_{n=1,3,5,\ldots} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Furthermore, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1,3,5,\ldots} \frac{1}{n^2} + \sum_{n=2,4,6,\ldots} \frac{1}{n^2} \\
&= \frac{\pi^2}{8} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\
&= \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}
\end{aligned}$$

which is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

as desired. □

2.6.9. Let $f(x) = \chi_{[a,b]}(x)$ be the characteristic function of the interval $[a, b] \subset [-\pi, \pi]$, that is,

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the Fourier series of f is given by

$$f(x) \sim \frac{b-a}{2\pi} + \sum_{n\neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi i n} e^{inx}.$$

The sum extends over all positive and negative integers excluding 0.

Solution. If $n \neq 0$, then we have

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\frac{2\pi i n x}{2\pi}}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_{[a,b]}(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{a} 0e^{-inx}\,dx + \int_{a}^{b} 1e^{-inx}\,dx + \int_{b}^{\pi} 0e^{-inx}\,dx\right) \\
&= \frac{1}{2\pi}\int_{a}^{b} e^{-inx}\,dx \\
&= -\frac{1}{2\pi i n}e^{-inx}\Big|_a^b \\
&= \frac{e^{-ina} - e^{-inb}}{2\pi i n}.
\end{aligned}$$

If $n = 0$, then we have

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\frac{2\pi i(0)x}{2\pi}}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_{[a,b]}(x)\,dx \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{a} \chi_{[a,b]}(x)\,dx + \int_{a}^{b} \chi_{[a,b]}(x)\,dx + \int_{b}^{\pi} \chi_{[a,b]}(x)\,dx\right) \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{a} 0\,dx + \int_{a}^{b} 1\,dx + \int_{b}^{\pi} 0\,dx\right) \\
&= \frac{1}{2\pi}(0 + (b-a) + 0) \\
&= \frac{b-a}{2\pi}.
\end{aligned}$$

So the Fourier series of f is given by

$$\begin{aligned}
f(x) &\sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n x}{2\pi}} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\
&= c_0 e^{in(0)} + \sum_{n\neq 0}^{\infty} c_n e^{inx} \\
&= \frac{b-a}{2\pi} + \sum_{n\neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi i n} e^{inx},
\end{aligned}$$

as desired. □

(b) Show that if $a \neq -\pi$ or $b \neq \pi$ and $a \neq b$, then the Fourier series does not converge absolutely for any x. [Hint: It suffices to prove that for many values of n one has $|\sin(n\theta_0)| \geq c > 0$ where $\theta_0 = \frac{b-a}{2}$.]

Solution. We have

$$
\begin{aligned}
|e^{-ina} - e^{-inb}| &= |e^{-inb}(e^{-ina}e^{inb} - 1)| \\
&= |e^{-inb}||e^{-ina}e^{inb} - 1| \\
&= 1 \cdot |e^{-ina+inb} - 1| \\
&= |e^{in(b-a)} - 1| \\
&= |\cos(n(b-a)) + i\sin(n(b-a)) - 1| \\
&= \sqrt{(\cos(n(b-a)) - 1)^2 + (\sin(n(b-a)))^2} \\
&= \sqrt{\cos^2(n(b-a)) - 2\cos(n(b-a)) + 1 + \sin^2(n(b-a))} \\
&= \sqrt{1 - 2\cos(n(b-a)) + 1} \\
&= \sqrt{2 - 2\cos(2n\theta_0)} \\
&= 2\left|\pm\sqrt{\frac{1-\cos(2n\theta_0)}{2}}\right| \\
&= 2|\sin(n\theta_0)|,
\end{aligned}
$$

Since we assume $a \neq -\pi$ or $b \neq \pi$ and $a \neq b$, it follows that the function $f(x) = \chi_{[a,b]}(x)$ is discontinuous on $[-\pi, \pi]$. Since the Fourier series must equal a discontinuous function, we must have $e^{ina} - e^{-inb} \neq 0$ (otherwise, $f(x) \sim \frac{b-a}{2}$ would be a constant function, which is of course continuous), which implies $|e^{ina} - e^{-inb}| > 0$. In fact, the above equality we calculated implies

$$
\begin{aligned}
|\sin(n\theta_0)| &= \frac{|e^{-ina} - e^{-inb}|}{2} \\
&> 0
\end{aligned}
$$

for all $n \neq 0$, and so there exists $c > 0$ that satisfies $|\sin(n\theta_0)| \geq c$ for many values of n on $[-\pi, \pi]$. So we conclude

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left|\frac{e^{-ina} - e^{-inb}}{2\pi in}e^{inx}\right| &= \sum_{n=1}^{\infty}\frac{|e^{-ina} - e^{-inb}|}{|2\pi in|}|e^{inx}| \\
&= \sum_{n=1}^{\infty}\frac{2|\sin(n\theta_0)|}{2\pi n} \cdot 1 \\
&= \sum_{n=1}^{\infty}\frac{|\sin(n\theta_0)|}{\pi n} \\
&\geq \sum_{\text{many } n \geq 1}\frac{c}{\pi n} \\
&= \frac{c}{\pi}\sum_{\text{many } n \geq 1}\frac{1}{n} \\
&= \infty,
\end{aligned}
$$

implying that the Fourier series of f does not converge absolutely. □

(c) However, prove that the Fourier series converges at every point x. What happens if $a = \pi$ or $b = -\pi$?

Solution. We can write

$$\begin{aligned}
\sum_{n\neq 0} \frac{e^{-ina}-e^{-inb}}{2\pi in} e^{inx} &= \sum_{n=-\infty}^{-1} \frac{e^{-ina}-e^{-inb}}{2\pi in} e^{inx} + \sum_{n=1}^{\infty} \frac{e^{-ina}-e^{-inb}}{2\pi in} e^{inx} \\
&= \sum_{n=1}^{\infty} \frac{e^{-i(-n)a}-e^{-i(-n)b}}{2\pi i(-n)} e^{i(-n)x} + \sum_{n=1}^{\infty} \frac{e^{-ina}-e^{-inb}}{2\pi in} e^{inx} \\
&= \sum_{n=1}^{\infty} -\frac{e^{ina}-e^{inb}}{2\pi in} e^{-inx} + \sum_{n=1}^{\infty} \frac{e^{-ina}-e^{-inb}}{2\pi in} e^{inx} \\
&= \sum_{n=1}^{\infty} \left(\frac{e^{-ina}-e^{-inb}}{2\pi in} e^{inx} - \frac{e^{ina}-e^{inb}}{2\pi in} e^{-inx} \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{e^{in(x-a)}-e^{in(x-b)}}{2\pi in} - \frac{e^{-in(x-a)}-e^{-in(x-b)}}{2\pi in} \right) \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{e^{in(x-a)}-e^{-in(x-a)}}{2i} - \frac{e^{in(x-b)}-e^{-in(x-b)}}{2i} \right) \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n(x-a)) - \sin(n(x-b))}{n} \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} a_n b_n,
\end{aligned}$$

provided that we define

$$\begin{aligned}
a_n &:= \frac{1}{n}, \\
b_n &:= \sin(n(x-a)) - \sin(n(x-b)).
\end{aligned}$$

Observe that the sequence $\{a_n\}_{n=1}^{\infty}$ decreases monotonically to 0, whereas $\{b_n\}_{n=1}^{\infty}$ is bounded, which implies

$$\begin{aligned}
\left| \sum_{n=1}^{N} b_n \right| &\leq \sum_{n=1}^{N} |b_n| \\
&= \sum_{n=1}^{N} |\sin(n(x-a)) - \sin(n(x-b))| \\
&\leq \sum_{n=1}^{N} |\sin(n(x-a))| + |\sin(n(x-b))| \\
&\leq \sum_{n=1}^{N} 1+1 \\
&= \sum_{n=1}^{N} 2 \\
&= 2N.
\end{aligned}$$

By Dirichlet's test (see Exercise 2.6.7(b) of the textbook), we conclude that

$$\sum_{n\neq 0} \frac{e^{-ina}-e^{-inb}}{2\pi in} e^{inx} = \frac{1}{\pi} \sum_{n=1}^{\infty} a_n b_n$$

converges for any $x \in \mathbb{R}$. This implies that the Fourier series of f converges for any $x \in \mathbb{R}$. □