MATH 147 discussion
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Winter 2021

Solutions to assigned homework problems from *Fourier Analysis: An Introduction* by Elias Stein and Rami Sakarchi

Homework 2

- Sect. 1.4: 1
- Sect. 3.3: 3, 4, 5, 7, 8, 9, 10, 12
- Sect. 3.4: 2, 3

1.4.1. We look for a solution of the steady-state heat equation $\Delta u = 0$ in the rectangle $R = \{(x, y) : 0 \le x \le \pi, 0 \le y \le 1\}$ that vanishes on the vertical sides of R, and so that

$$u(x, 0) = f_0(x) \text{ and } u(x, 1) = f_1(x),$$

where f_0 and f_1 are initial data which fix the temperature distribution on the horizontal sides of the rectangle. Use separation of variables to show that if f_0 and f_1 have Fourier expansions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin(kx) \text{ and } f_1(x) = \sum_{k=1}^{\infty} B_k \sin(kx)$$

then

$$u(x, y) = \sum_{k=1}^{\infty} \left(\frac{\sinh(k(1-y))}{\sinh(k)} A_k + \frac{\sinh(ky)}{\sinh(k)} B_k \right) \sinh(kx).$$

We recall the definitions of the hyperbolic sine and cosine functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Solution. To commence the method of separation of variables, write

$$u(x, y) = \varphi(x)\psi(y),$$

as suggested by page 4 of the textbook. Our partial derivatives are

$$u_{xx}(x, y) = \varphi'(x)\psi(y),$$
$$u_{yy}(x, y) = \varphi(x)\psi''(y)$$

So the steady-state heat equation $\Delta u = 0$, or $u_{xx} + u_{yy} = 0$, becomes

$$\varphi_{xx}(x)\psi(y) + \varphi(x)\psi_{yy}(y) = 0.,$$

which we can algebraically rearrange to write

$$\frac{\varphi_{xx}}{\varphi} = -\frac{\psi_{yy}}{\psi} = -\lambda,$$

where $\lambda \in \mathbb{R}$ is a constant in both x and y. This produces the system of two ordinary differential equations

$$\varphi_{xx} + \lambda\varphi = 0,$$
$$\psi_{yy} - \lambda\psi = 0.$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$\varphi(x) = \begin{cases} C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0, \\ C_1 x + C_2 & \text{if } \lambda = 0, \\ C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) & \text{if } \lambda > 0, \end{cases}$$

$$\psi(y) = \begin{cases} D_1 \cos(\sqrt{-\lambda}y) + D_2 \sin(\sqrt{-\lambda}y) & \text{if } \lambda < 0, \\ D_1 y + D_2 & \text{if } \lambda = 0, \\ D_1 e^{\sqrt{\lambda}y} + D_2 e^{-\sqrt{\lambda}y} & \text{if } \lambda > 0, \end{cases}$$

where C_1, C_2, D_1, D_2 are constants. Now, the boundary conditions

$$u(0, y) = u(\pi, y) = 0$$

are equivalent to

$$\varphi(0)\psi(y) = 0,$$
$$\varphi(\pi)\psi(y) = 0,$$

which imply either $\psi(y) = 0$ or $\varphi(0) = \varphi(\pi) = 0$. If $\psi(y) = 0$, then we would have

$$\begin{aligned} u(x, y) &= \varphi(x)\psi(y) \\ &= \varphi(x)0 \\ &= 0, \end{aligned}$$

which would be a trivial solution. So we should assume

$$\varphi(0) = \varphi(\pi) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then we have

  $$\begin{aligned} &\varphi(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}, \\ &\varphi(0) = 0, \end{aligned}$$

  which implies $C_1 + C_2 = 0$, or $C_2 = -C_1$. So we have

  $$\begin{aligned} \varphi(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \\ &= C_1 e^{\sqrt{-\lambda}x} - C_1 e^{-\sqrt{-\lambda}x} \\ &= C_1 (e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}). \end{aligned}$$

  We notice $e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi} \neq 0$ unless $\lambda = 0$. This means

  $$\begin{aligned} &\varphi(x) = C_1 (e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}), \\ &\varphi(\pi) = 0 \end{aligned}$$

  implies $C_1 = 0$, and so we have

  $$\begin{aligned} \varphi(x) &= C_1 (e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}) \\ &= 0(e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}) \\ &= 0, \end{aligned}$$

  which would mean u is a trivial solution. Therefore, the problem has no negative eigenvalues.
- Case 2: Suppose $\lambda = 0$. Then we have

  $$\begin{aligned} &\varphi(x) = C_1 x + C_2, \\ &\varphi(0) = 0, \end{aligned}$$

  implies $C_2 = 0$, and so we have

  $$\begin{aligned} \varphi(x) &= C_1 x + C_2 \\ &= C_1 x + 0 \\ &= C_1 x. \end{aligned}$$

  Furthermore, $\varphi(\pi) = 0$ implies $C_1 = 0$, and so we write $\varphi(x) = 0$. Therefore, we have

  $$\begin{aligned} u_0(x, y) &= \varphi(x)\psi(y) \\ &= 0\psi(y) \\ &= 0, \end{aligned}$$

  which is a trivial solution.
- Case 3: Suppose $\lambda > 0$. Then we have

  $$\begin{aligned} &\varphi(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x), \\ &\varphi(0) = 0, \end{aligned}$$

which implies $C_1 = 0$, and so we have

$$\begin{aligned}\varphi(x) &= C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \\ &= 0 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \\ &= C_2 \sin(\sqrt{\lambda}x).\end{aligned}$$

Next, we have

$$\begin{aligned}&\varphi(x) = C_2 \sin(\sqrt{\lambda}x), \\ &\varphi(\pi) = 0\end{aligned}$$

implies either $C_2 = 0$ or $\sin(\sqrt{\lambda}\pi) = 0$. But $C_2 = 0$ (with $C_1 = 0$) implies $\varphi(x) = 0$ and that $u(x, y)$ would be a trivial solution. So we should assume $\sqrt{\lambda}\pi = n\pi$, or equivalently the eigenvalues

$$\lambda_k = \lambda = n^2,$$

with the corresponding eigenfunctions

$$\begin{aligned}\varphi_k(x) &= C_{2,k} \sin(\sqrt{\lambda_k}x) \\ &= C_{2,k} \sin(\sqrt{n^2}x) \\ &= C_{2,k} \sin(kx).\end{aligned}$$

Next, we need to express ψ_k as a linear combination of hyperbolic sine functions. We can first rewrite

$$\begin{aligned}\psi_k(y) &= D_{1,k}e^{ky} + D_{2,k}e^{-ky} \\ &= 2D_{1,k}\left(\frac{e^{ky} - e^{-ky}}{2} + \frac{e^{-ky}}{2}\right) - 2e^{-k}D_{2,k}\left(\frac{e^{n(y-1)} - e^{-k(y-1)}}{2} - \frac{e^{n(y-1)}}{2}\right) \\ &= 2D_{1,k}\left(\sinh(ky) + \frac{e^{-ky}}{2}\right) - 2e^{-k}D_{2,k}\left(\sinh(k(y-1)) - \frac{e^{n(y-1)}}{2}\right) \\ &= 2D_{1,k} \sinh(ky) + D_{1,k}e^{-ky} - 2e^{-k}D_{2,k} \sinh(k(y-1)) + e^{-2k}D_{2,k}e^{ky} \\ &= 2D_{1,k} \sinh(ky) - 2e^{-k}D_{2,k} \sinh(k(y-1)) + e^{-2k}D_{2,k}e^{ky} + D_{1,k}e^{-ky}.\end{aligned}$$

As the choice of constants is arbitrary, we are allowed to relabel the constants. By relabeling the constants, we can write

$$\begin{aligned}\psi_k(y) &= D_{1,k} \sinh(ky) + D_{2,k} \sinh(k(y-1)) + D_{3,k}e^{ky} - D_{3,k}e^{-ky} \\ &= D_{1,k} \sinh(ky) + D_{2,k} \sinh(k(y-1)) + 2D_{3,k}\frac{e^{-ky} - e^{ky}}{2} \\ &= D_{1,k} \sinh(ky) + D_{2,k} \sinh(k(y-1)) + 2D_{3,k} \sinh(ky) \\ &= (D_{1,k} + 2D_{3,k}) \sinh(ky) + D_{2,k} \sinh(k(y-1)).\end{aligned}$$

By relabeling the constants one more time, we can finally write

$$\psi_k(y) = D_{1,k} \sinh(ky) + D_{2,k} \sinh(k(y-1)).$$

Therefore, if we write $a_k := C_{2,k}D_{1,k}$ and $b_k := C_{2,k}D_{2,k}$, then we have

$$\begin{aligned}u_k(x, y) &= X_k(x)\psi_k(y) \\ &= (C_{2,k} \sin(kx))(D_{1,k} \sinh(ky) + D_{2,k} \sinh(k(y-1))) \\ &= \sin(kx)(C_{2,k}D_{1,k} \sinh(ky) + C_{2,k}D_{2,k} \sinh(k(y-1))) \\ &= \sin(kx)(a_k \sinh(ky) + b_k \sinh(k(y-1))).\end{aligned}$$

for $k = 1, 2, 3, \ldots$, which is a nontrivial solution.

Given

$$u(x, y) = \sum_{k=1}^{\infty} \sin(kx)(a_k \sinh(ky) + b_k \sinh(k(y-1))),$$

we have

$$\begin{aligned}f_0(x) = u(x, 0) &= \sum_{k=1}^{\infty} b_k \sin(kx) \sinh(-k), \\ f_1(x) = u(x, 1) &= \sum_{k=1}^{\infty} a_k \sin(kx) \sinh(k).\end{aligned}$$

Now, recall

$$\int_0^\pi \sin(kx)\sin(lx)\,dx = \begin{cases} \frac{\pi}{2} & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

Consequently, the Fourier sine series expansion of f_0 and f_1 suggest that A_k and B_k are the Fourier sine coefficients of f_0 and f_1, respectively. So we obtain

$$\begin{aligned} A_k &= \frac{2}{\pi}\int_0^\pi f_0(x)\sin(kx)\,dx \\ &= \frac{2}{\pi}\int_0^\pi u(x,0)\sin(kx)\,dx \\ &= \frac{2}{\pi}\int_0^\pi \left(\sum_{l=1}^\infty \sin(lx)b_l\sin(lx)\sinh(-m)\right)\sin(kx)\,dx \\ &= \frac{2}{\pi}\sum_{l=1}^\infty b_l\sinh(-m)\int_0^\pi \sin(lx)\sin(kx)\,dx \\ &= \frac{2}{\pi}b_k\sinh(-k)\frac{\pi}{2} \\ &= -\frac{2}{\pi}b_k\sinh(k)\frac{\pi}{2} \\ &= -b_k\sinh(k) \end{aligned}$$

and

$$\begin{aligned} B_k &= \frac{2}{\pi}\int_0^\pi f_1(x)\sin(kx)\,dx \\ &= \frac{2}{\pi}\int_0^\pi u(x,1)\sin(kx)\,dx \\ &= \frac{2}{\pi}\int_0^\pi \left(\sum_{l=1}^\infty a_l\sin(lx)\sinh(l)\right)\sin(kx)\,dx \\ &= \frac{2}{\pi}\sum_{l=1}^\infty a_l\sinh(l)\int_0^\pi \sin(lx)\sin(kx)\,dx \\ &= \frac{2}{\pi}a_k\sinh(k)\frac{\pi}{2} \\ &= a_k\sinh(k). \end{aligned}$$

So we obtain the coefficients

$$\begin{aligned} a_k &= \frac{B_k}{\sinh(k)}, \\ b_k &= -\frac{A_k}{\sinh(k)}. \end{aligned}$$

So our formal solution is

$$\begin{aligned} u(x,y) &= \sum_{k=1}^\infty \sin(kx)(a_k\sinh(ky) + b_k\sinh(k(y-1))) \\ &= \sum_{k=1}^\infty \sin(kx)\left(\frac{B_k}{\sinh(k)}\sinh(ky) - \frac{A_k}{\sinh(k)}\sinh(k(y-1))\right) \\ &= \sum_{k=1}^\infty \left(-\frac{\sinh(k(y-1))}{\sinh(k)}A_k + \frac{\sinh(ky)}{\sinh(k)}B_k\right)\sin(kx) \\ &= \sum_{k=1}^\infty \left(\frac{\sinh(k(1-y))}{\sinh(k)}A_k + \frac{\sinh(ky)}{\sinh(k)}B_k\right)\sin(kx), \end{aligned}$$

as desired. □

3.3.3. Construct a sequence of integrable functions $\{f_k\}$ on $[0, 2\pi]$ such that

$$\lim_{k\to\infty}\frac{1}{2\pi}\int_0^{2\pi}|f_k(\theta)|^2\,d\theta = 0$$

but $\lim_{k\to\infty} f_k(\theta)$ fails to exist for any θ.

[Hint: Choose a sequence of intervals $I_k \subset [0, 2\pi]$ whose lengths tend to 0, and so that each point belongs to infinitely many of them; then let $f_k = \chi_{I_k}$.]

Solution. Consider a sqeuence $\{I_k\}_{k=1}^{\infty}$ defined by $I_k := [0, \frac{1}{k}]$, which satisfies $I_k \subset [0, 2\pi]$, so that their lengths $|I_k|$ tend to 0 as $k \to \infty$. If we choose $f_k = \chi_{I_k}$, then we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |f_k(\theta)|^2\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} |\chi_{I_k}(\theta)|^2\, d\theta \\
&= \frac{1}{2\pi}\int_0^{\frac{1}{k}} |\chi_{I_k}(\theta)|^2\, d\theta + \frac{1}{2\pi}\int_{\frac{1}{k}}^{2\pi} |\chi_{I_k}(\theta)|^2\, d\theta \\
&= \frac{1}{2\pi}\int_0^{\frac{1}{k}} 1^2\, d\theta + \frac{1}{2\pi}\int_{\frac{1}{k}}^{2\pi} 0^2\, d\theta \\
&= \frac{1}{2\pi}\int_0^{\frac{1}{k}} 1\, d\theta \\
&= \frac{1}{2\pi}\theta\Big|_0^{\frac{1}{k}} \\
&= \frac{1}{2\pi}\left(\frac{1}{k} - 0\right) \\
&= \frac{1}{2\pi k},
\end{aligned}$$

which implies

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |f_k(\theta)|^2\, d\theta &= \lim_{k\to\infty} \frac{1}{2\pi k} \\
&= \frac{1}{2\pi}\lim_{k\to\infty}\frac{1}{k} \\
&= \frac{1}{2\pi}(0) \\
&= 0.
\end{aligned}$$

Now, we will show that $\lim_{k\to\infty} f_k(\theta)$ fails to exist for any θ. At the same time, we also have

$$\lim_{k\to\infty} \chi_{I_k}(\theta) = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases}$$

which is a Dirac delta distribution, not a function. In other words, there does not exist a function f that is a limit of $\{f_k\}_{k=1}^{\infty}$. □

3.3.4. Recall the vector space $\mathcal{R}$ of integrable functions, with its inner product and norm

$$\|f\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\, dx\right)^{\frac{1}{2}}.$$

(a) Show that there exist non-zero integrable functions f for which $\|f\| = 0$.

Solution. Choose for instance

$$f(x) = \begin{cases} 0 & \text{if } x \neq \pi, \\ 1 & \text{if } x = \pi. \end{cases}$$

Then we have $f \in \mathcal{R}$ and f is nonzero, and

$$\begin{aligned}
\|f\|^2 &= \frac{1}{2\pi}\int_0^{2\pi} |f(x)|\, dx \\
&= \frac{1}{2\pi}\left(\int_0^{\pi} |f(x)|\, dx + \int_{\pi}^{2\pi} |f(x)|\, dx\right) \\
&= \frac{1}{2\pi}(0 + 0) \\
&= 0,
\end{aligned}$$

meaning that f satisfies all the requested properties. □

(b) However, show that if $f \in \mathcal{R}$ with $\|f\| = 0$, then $f(x) = 0$ whenever f is continuous at x.

Solution. Suppose instead $f(x) \neq 0$ and f is continuous at x for all $0 \leq x \leq 2\pi$. Then we have $f(x) > 0$ or $f(x) < 0$ for all $0 \leq x \leq 2\pi$. In either case, we have $|f(x)| > 0$, which implies $|f(x)|^2 > 0^2 = 0$ for all $0 \leq x \leq 2\pi$, and so we obtain

$$\begin{aligned} \|f\|^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx \\ &> \frac{1}{2\pi} \int_0^{2\pi} 0 \, dx \\ &= 0, \end{aligned}$$

or equivalently $\|f\| > 0$, which contradicts the assumption $\|f\| = 0$. So we are forced to conclude $f(x) = 0$. □

(c) Conversely, show that if $f \in \mathcal{R}$ vanishes at all of its points of continuity, then $\|f\| = 0$.

Solution. Since we assume $f \in \mathcal{R}$, it follows by Theorem 1.7 of the Appendix (Integration) in Stein and Shakarchi that f is continuous on $0 \leq x \leq 2\pi$ except on a set of measure zero. We also assume that f vanishes at all of its points of continuity; in this case, we have $f = 0$ except on a set of measure zero. Let $A \subset [0, 2\pi]$ be such a set of measure zero; that is, A satisfies $|A| = 0$, where $|A|$ denotes the length of A. Then we have $f(x) = 0$ for all $x \in [0, 2\pi] \setminus A$. Note that a set of measure zero can be either empty or nonempty. If A is nonempty, then we have

$$\begin{aligned} 0 &\leq \int_A |f(x)|^2 \, dx \\ &\leq \int_A \sup_{x \in A} |f(x)|^2 \, dx \\ &= \sup_{x \in A} |f(x)|^2 \int_A 1 \, dx \\ &= \sup_{x \in A} |f(x)|^2 |A| \\ &= \sup_{x \in A} |f(x)|^2 \cdot 0 \\ &= 0, \end{aligned}$$

which implies

$$\int_A |f(x)|^2 \, dx = 0.$$

So we obtain

$$\begin{aligned} \|f\|^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx \\ &= \frac{1}{2\pi} \left(\int_A |f(x)|^2 \, dx + \int_{[0,2\pi] \setminus A} |f(x)|^2 \, dx \right) \\ &= \frac{1}{2\pi} \left(0 + \int_{[0,2\pi] \setminus A} 0^2 \, dx \right) \\ &= 0, \end{aligned}$$

which is equivalent to $\|f\| = 0$, as desired. On the other hand, if A is empty, or $A = \varnothing$, then the argument is somewhat trivial: we have $|A| = |\varnothing| = 0$ and $f(x) = 0$ for all $0 \leq x \leq 2\pi$, and so we obtain

$$\begin{aligned} \|f\|^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} 0^2 \, dx \\ &= 0 \end{aligned}$$

which is equivalent to $\|f\| = 0$, as desired. □

3.3.5. Let

$$f(\theta) = \begin{cases} 0 & \text{for } \theta = 0, \\ \log(\frac{1}{\theta}) & \text{for } 0 < \theta \leq 2\pi, \end{cases}$$

and define a sequence of functions in $\mathcal{R}$ by

$$b_n(\theta) = \begin{cases} 0 & \text{for } 0 \leq \theta \leq \frac{1}{n}, \\ f(\theta) & \text{for } \frac{1}{n} < \theta \leq 2\pi. \end{cases}$$

Prove that $\{b_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{R}$. However, f does not belong to $\mathcal{R}$.

Solution. Since we have $f(\theta) = \log(\frac{1}{\theta})$ for $0 < \theta \leq 2\pi$, this holds true in particular for $\frac{1}{n} < \theta \leq 2\pi$ for all $n = 1, 2, 3, \ldots$. So we can actually write

$$f_n(\theta) = \begin{cases} 0 & \text{for } 0 \leq \theta \leq \frac{1}{n}, \\ \log(\frac{1}{\theta}) & \text{for } \frac{1}{n} < \theta \leq 2\pi. \end{cases}$$

Now we will show that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence with respect to the norm of $\mathcal{R}$. Let m, n be large positive integers with the assumption $m > n$ without loss of generality. We apply the Pythagorean Theorem for the norm of $\mathcal{R}$ in order to obtain

$$\begin{aligned}
\|f_n - f_m\|^2 &= \|(f_n - f_m) + f_m\|^2 - \|f_m\|^2 \\
&= \|f_n\|^2 - \|f_m\|^2 \\
&= \frac{1}{2\pi}\int_0^{2\pi} |b_n(\theta)|^2 \, d\theta - \frac{1}{2\pi}\int_0^{2\pi} |f_m(\theta)|^2 \, d\theta \\
&= \frac{1}{2\pi}\left(\int_0^{2\pi} |b_n(\theta)|^2 \, d\theta - \int_0^{2\pi} |f_m(\theta)|^2 \, d\theta\right) \\
&= \frac{1}{2\pi}\left(\int_{\frac{1}{n}}^{2\pi} \left|\log\left(\frac{1}{\theta}\right)\right|^2 \, d\theta - \int_{\frac{1}{m}}^{2\pi} \left|\log\left(\frac{1}{\theta}\right)\right|^2 \, d\theta\right) \\
&= \frac{1}{2\pi}\int_{\frac{1}{n}}^{\frac{1}{m}} \left|\log\left(\frac{1}{\theta}\right)\right|^2 \, d\theta \\
&= \frac{1}{2\pi}\int_{\frac{1}{n}}^{\frac{1}{m}} (\log(\theta))^2 \, d\theta \\
&= \theta(\log(\theta))^2\Big|_{\frac{1}{n}}^{\frac{1}{m}} - 2\int_{\frac{1}{n}}^{\frac{1}{m}} \log(\theta) \, d\theta \\
&= \frac{1}{m}\left(\log\left(\frac{1}{m}\right)\right)^2 - \frac{1}{n}\left(\log\left(\frac{1}{n}\right)\right)^2 - 2\left(\theta\log(\theta)\Big|_{\frac{1}{n}}^{\frac{1}{m}} - \int_{\frac{1}{n}}^{\frac{1}{m}} 1 \, d\theta\right) \\
&= \frac{1}{m}\left(\log\left(\frac{1}{m}\right)\right)^2 - \frac{1}{n}\left(\log\left(\frac{1}{n}\right)\right)^2 - 2\theta\log(\theta)\Big|_{\frac{1}{n}}^{\frac{1}{m}} + 2\theta\Big|_{\frac{1}{n}}^{\frac{1}{m}} \\
&= \frac{1}{m}\left(\log\left(\frac{1}{m}\right)\right)^2 - \frac{1}{n}\left(\log\left(\frac{1}{n}\right)\right)^2 - \frac{2}{m}\log\left(\frac{1}{m}\right) + \frac{2}{n}\log\left(\frac{1}{n}\right) + \frac{2}{m} - \frac{2}{n} \\
&= \frac{1}{m}\left(\left(\log\left(\frac{1}{m}\right)\right)^2 - 2\log\left(\frac{1}{m}\right) + 2\right) - \frac{1}{n}\left(\left(\log\left(\frac{1}{n}\right)\right)^2 - 2\log\left(\frac{1}{n}\right) + 2\right) \\
&= \frac{1}{m}\left(\left(\log\left(\frac{1}{m}\right) - 1\right)^2 + 1\right) - \frac{1}{n}\left(\left(\log\left(\frac{1}{n}\right) - 1\right)^2 + 1\right) \\
&= \frac{(\log(m) + 1)^2 + 1}{m} - \frac{(\log(n) + 1)^2 + 1}{n} \\
&\to 0 - 0 \\
&= 0
\end{aligned}$$

as $m, n \to \infty$, which signifies that $\{b_n\}_{n=1}^{\infty}$ is a Cauchy sequence. The convergence towards the end of our previous calculations is due to the following limit (for my method, I applied l'Hôpital's rule twice as follows):

$$\begin{aligned}
\lim_{x\to\infty} \frac{(\log(x) + 1)^2 + 1}{x} &= \lim_{x\to\infty} \frac{\frac{d}{dx}((\log(x) + 1)^2 + 1)}{\frac{d}{dx}x} \\
&= \lim_{x\to\infty} \frac{2(\log(x) + 1)\frac{1}{x}}{1} \\
&= \lim_{x\to\infty} \frac{2(\log(x) + 1)}{x} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}2(\log(x) + 1)}{\frac{d}{dx}x} \\
&= \lim_{x\to\infty} \frac{\frac{2}{x}}{1} \\
&= \lim_{x\to\infty} \frac{2}{x} \\
&= 0,
\end{aligned}$$

as desired. ◻

3.3.7. Show that the trigonometric series

$$\sum_{n=2}^{\infty} \frac{1}{\log(n)} \sin(nx)$$

converges for every x, yet it is not the Fourier series of a Riemannian integrable function.

Solution. We can write

$$\sum_{n=2}^{\infty} \frac{1}{\log(n)} \sin(nx) = \sum_{n=2}^{\infty} a_n b_n,$$

provided that we define

$$\begin{aligned} a_n &:= \frac{1}{\log(n)}, \\ b_n &:= \sin(nx). \end{aligned}$$

Observe that the sequence $\{a_n\}_{n=1}^{\infty}$ decreases monotonically to 0, whereas $\{b_n\}_{n=1}^{\infty}$ is bounded, which implies

$$\begin{aligned} \left| \sum_{n=1}^{N} b_n \right| &\leq \sum_{n=1}^{N} |b_n| \\ &= \sum_{n=1}^{N} |\sin(nx)| \\ &= \sum_{n=1}^{N} 1 \\ &= N. \end{aligned}$$

By Dirichlet's test (see Exercise 2.6.7(b) of the textbook), we conclude that

$$\sum_{n=2}^{\infty} \frac{1}{\log(n)} \sin(nx) = \sum_{n=2}^{\infty} a_n b_n,$$

converges for any $x \in \mathbb{R}$. Consider some function f whose Fourier series is

$$\sum_{n=2}^{\infty} c_n \sin(nx),$$

where we define $c_n := \frac{1}{\log(n)}$. Then by Parseval's identity, we have

$$\begin{aligned} \|f\|^2 &= \sum_{n=2}^{\infty} |c_n|^2 \\ &= \sum_{n=2}^{\infty} \frac{1}{|\log(n)|^2} \\ &= \infty, \end{aligned}$$

meaning that f is not Riemann integrable. There are many ways to show that the series

$$\sum_{n=2}^{\infty} \frac{1}{|\log(n)|^2}$$

is divergent. Perhaps the most elementary method of showing this is the integral test: we have

$$\begin{aligned} \int_2^{\infty} \frac{1}{|\log(x)|^2}\, dx &= \int_2^{\infty} \frac{1}{\log(x)\log(x)}\, dx \\ &\geq \int_2^{\infty} \frac{1}{x\log(x)}\, dx \\ &= \int_{\log(2)}^{\infty} \frac{1}{u}\, du \\ &= |\log(u)|\Big|_{\log(2)}^{\infty} \\ &= \log(\infty) - \log(\log(2)) \\ &= \infty. \end{aligned}$$

Therefore, the series in question diverges by the integral test. □

3.3.8. Exercise 6 in Chapter 2 dealt with the sums

$$\sum_{n=1,3,5,\ldots} \frac{1}{n^2} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Similar sums can be derived using the methods of this chapter.

(a) Let f be the functiondefined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$. Use Parseval's identity to find the sums of the following two series:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution. We have already computed in Exercise 2.6.6 that the n^{th} Fourier coefficient of $f(\theta) = |\theta|$ is

$$c_n = \begin{cases} \frac{\pi}{2} & \text{if } n = 0, \\ \frac{-1+(-1)^n}{\pi n^2} & \text{if } n \neq 0 \end{cases}$$

for all integers n. We have

$$\begin{aligned} \|f\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\theta|^2 \, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta^2 \, d\theta \\ &= \frac{1}{2\pi} \left. \frac{\theta^3}{3} \right|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \frac{\pi^3 - (-\pi)^3}{3} \\ &= \frac{1}{2\pi} \frac{2\pi^3}{3} \\ &= \frac{\pi^2}{3} \end{aligned}$$

and, by Parseval's identity,

$$\begin{aligned} \|f\|^2 &= \sum_{n=-\infty}^{\infty} |c_n|^2 \\ &= |c_0|^2 + \sum_{\substack{n \neq 0 \\ n \in \mathbb{Z}}} |c_n|^2 \\ &= \left| \frac{\pi}{2} \right|^2 + \sum_{\substack{n \neq 0 \\ n \in \mathbb{Z}}} \left| \frac{-1 + (-1)^n}{\pi n^2} \right|^2 \\ &= \frac{\pi^2}{4} + 2 \sum_{n=1}^{\infty} \frac{|-1 + (-1)^n|^2}{\pi^2 n^4} \\ &= \frac{\pi^2}{4} + 2 \left(\sum_{n=1,3,5,\ldots} \frac{|-1 + (-1)^n|^2}{\pi^2 n^4} + \sum_{n=2,4,6,\ldots} \frac{|-1 + (-1)^n|^2}{\pi^2 n^4} \right) \\ &= \frac{\pi^2}{4} + 2 \left(\sum_{n=1,3,5,\ldots} \frac{|-2|^2}{\pi^2 n^4} + \sum_{n=2,4,6,\ldots} \frac{|0|^2}{\pi^2 n^4} \right) \\ &= \frac{\pi^2}{4} + 2 \sum_{n=1,3,5,\ldots} \frac{4}{\pi^2 n^4} \\ &= \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{1}{(2n+1)^4} \\ &= \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}. \end{aligned}$$

We combine our two expressions of $\|f\|^2$ to conclude

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4},$$

which is algebraically equivalent to

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} &= \frac{\pi^2}{8}\left(\frac{\pi^2}{3} - \frac{\pi^2}{4}\right) \\ &= \frac{\pi^2}{8}\frac{\pi^2}{12} \\ &= \frac{\pi^4}{96},\end{aligned}$$

which is the first sum. Furthermore, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^4} &= \sum_{n=1,3,5,\ldots} \frac{1}{n^4} + \sum_{n=2,4,6,\ldots} \frac{1}{n^4} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} + \sum_{n=1}^{\infty} \frac{1}{(2n)^4} \\ &= \frac{\pi^4}{96} + \frac{1}{16}\sum_{n=1}^{\infty} \frac{1}{n^4},\end{aligned}$$

which is algebraically equivalent to

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{16}{15}\frac{\pi^4}{96} \\ &= \frac{\pi^4}{90},\end{aligned}$$

which is the second sum. □

(b) Consider the 2π-periodic odd functiondefined on $[0,\pi]$ by $f(\theta) = \theta(\pi - \theta)$. Show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Solution. We have already computed in Exercise 2.6.4 that the n^{th} Fourier coefficient of $f(\theta) = \theta(\pi - \theta)$ is

$$c_n = \begin{cases} 0 & \text{if } n = 0, \\ \frac{2((-1)^n - 1)}{\pi n^3} i & \text{if } n \neq 0 \end{cases}$$

for all integers n. We have

$$\begin{aligned}\|f\|^2 &= \frac{1}{\pi}\int_0^{\pi} |f(\theta)|^2 \, d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \theta^2(\pi - \theta)^2 \, d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \theta^2(\pi^2 - 2\pi\theta + \theta^2) \, d\theta \\ &= \frac{1}{\pi}\left(\int_0^{\pi} \theta^2(\pi^2 - 2\pi\theta + \theta^2) \, d\theta\right) \\ &= \frac{1}{\pi}\left(\int_0^{\pi} \pi^2\theta^2 - 2\pi\theta^3 + \theta^4 \, d\theta\right) \\ &= \frac{1}{\pi}\left(\pi^2\int_0^{\pi} \theta^2 \, d\theta - 2\pi\int_0^{\pi} \theta^3 \, d\theta + \int_0^{\pi} \theta^4 \, d\theta\right) \\ &= \frac{1}{\pi}\left(\pi^2 \frac{\theta^3}{3}\bigg|_0^{\pi} - 2\pi\frac{\theta^4}{4}\bigg|_0^{\pi} + \frac{\theta^5}{5}\bigg|_0^{\pi}\right) \\ &= \frac{1}{\pi}\left(\pi^2\frac{\pi^3 - 0^3}{3} - 2\pi\frac{\pi^4 - 0^4}{4} + \frac{\pi^5 - 0^5}{5}\right) \\ &= \frac{1}{\pi}\left(\frac{\pi^5}{3} - \frac{\pi^5}{2} + \frac{\pi^5}{5}\right) \\ &= \frac{\pi^4}{30}\end{aligned}$$

and, by Parseval's identity,

$$\begin{aligned}
\|f\|^2 &= \sum_{n=-\infty}^{\infty} |c_n|^2 \\
&= |c_0|^2 + \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} |c_n|^2 \\
&= |0|^2 + \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \left|\frac{2((-1)^n - 1)}{\pi n^3} i\right|^2 \\
&= 2\sum_{n=1}^{\infty} \frac{|2((-1)^n - 1)|^2}{\pi^2 n^6} \\
&= 8\sum_{n=1}^{\infty} \frac{|(-1)^n - 1|^2}{\pi^2 n^6} \\
&= 8\left(\sum_{n=1,3,5,\ldots} \frac{|(-1)^n - 1|^2}{\pi^2 n^6} + \sum_{n=2,4,6,\ldots} \frac{|(-1)^n - 1|^2}{\pi^2 n^6}\right) \\
&= 8\left(\sum_{n=1,3,5,\ldots} \frac{|-2|^2}{\pi^2 n^6} + \sum_{n=2,4,6,\ldots} \frac{|0|^2}{\pi^2 n^6}\right) \\
&= 8\sum_{n=1,3,5,\ldots} \frac{4}{\pi^2 n^6} \\
&= \frac{32}{\pi^2}\sum_{n=1,3,5,\ldots} \frac{1}{(2n+1)^6} \\
&= \frac{32}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6}.
\end{aligned}$$

We combine our two expressions of $\|f\|^2$ to conclude

$$\frac{\pi^4}{5} = \frac{32}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6},$$

which is algebraically equivalent to

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6} &= \frac{\pi^2}{32}\frac{\pi^4}{30} \\
&= \frac{\pi^6}{960},
\end{aligned}$$

which is the first sum. Furthermore, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^6} &= \sum_{n=1,3,5,\ldots}\frac{1}{n^6} + \sum_{n=2,4,6,\ldots}\frac{1}{n^6} \\
&= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^6} + \sum_{n=1}^{\infty}\frac{1}{(2n)^6} \\
&= \frac{\pi^6}{960} + \frac{1}{64}\sum_{n=1}^{\infty}\frac{1}{n^6},
\end{aligned}$$

which is algebraically equivalent to

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{64}{63}\frac{\pi^6}{960} \\
&= \frac{\pi^6}{945},
\end{aligned}$$

which is the second sum. □

3.3.9. Show that, if α is not an integer, the Fourier series of

$$f(x) = \frac{\pi}{\sin(\pi\alpha)} e^{i(\pi - x)\alpha}$$

on $[0, 2\pi]$ is given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n+\alpha}.$$

Apply Parseval's formula to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)}.$$

Solution. For all integers n, the Fourier coefficient is

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\pi}{\sin(\pi\alpha)} e^{i(\pi-x)\alpha}e^{-inx}\,dx \\
&= \frac{e^{i\pi\alpha}}{2\sin(\pi\alpha)}\int_0^{2\pi} e^{-i(n+\alpha)x}\,dx \\
&= \frac{e^{i\pi\alpha}}{2\sin(\pi\alpha)}\left(-\frac{1}{i(n+\alpha)}e^{-i(n+\alpha)x}\bigg|_0^{2\pi}\right) \\
&= -\frac{e^{i\pi\alpha}}{2\sin(\pi\alpha)}\frac{e^{-i(n+\alpha)2\pi}-1}{i(n+\alpha)} \\
&= -\frac{e^{i\pi\alpha}}{2\sin(\pi\alpha)}\frac{e^{-2i\pi\alpha}e^{2i\pi n}-1}{i(n+\alpha)} \\
&= -\frac{e^{i\pi\alpha}}{2\sin(\pi\alpha)}\frac{e^{-2i\pi\alpha}-1}{i(n+\alpha)} \\
&= \frac{1}{(n+\alpha)\sin(\pi\alpha)}\frac{e^{i\pi\alpha}-e^{-i\pi\alpha}}{2i} \\
&= \frac{\sin(\pi\alpha)}{(n+\alpha)\sin(\pi\alpha)} \\
&= \frac{1}{n+\alpha},
\end{aligned}$$

and so the Fourier series on $[2, \pi]$ is given by

$$\begin{aligned}
f(x) &\sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \\
&= \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n+\alpha}.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\|f\|^2 &= \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{\pi}{\sin(\pi\alpha)}e^{i(\pi-x)\alpha}\right|^2\,dx \\
&= \frac{1}{2\pi}\frac{\pi^2}{\sin^2(\pi\alpha)}\int_0^{2\pi} |e^{i(\pi-x)\alpha}|^2\,dx \\
&= \frac{1}{2\pi}\frac{\pi^2}{\sin^2(\pi\alpha)}\int_0^{2\pi} 1^2\,dx \\
&= \frac{1}{2\pi}\frac{\pi^2}{\sin^2(\pi\alpha)}\int_0^{2\pi} 1\,dx \\
&= \frac{1}{2\pi}\frac{\pi^2}{\sin^2(\pi\alpha)}x\bigg|_0^{2\pi} \\
&= \frac{1}{2\pi}\frac{\pi^2}{\sin^2(\pi\alpha)}(2\pi-0) \\
&= \frac{\pi^2}{\sin^2(\pi\alpha)}
\end{aligned}$$

and, by Parseval's identity,

$$\begin{aligned}
\|f\|^2 &= \sum_{n=-\infty}^{\infty} |c_n|^2 \\
&= \sum_{n=-\infty}^{\infty} \left|\frac{1}{n+\alpha}\right|^2 \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2}.
\end{aligned}$$

Equate our two expressions of $\|f\|^2$ together to conclude

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)},$$

as desired. □

3.3.10. Consider the example of a vibrating string which we analyzed in Chapter 1. The displacement $u(x,t)$ of the string at time t satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

where $c^2 = \frac{\tau}{\rho}$. The string is subject to the initial conditions

$$u(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = g(x),$$

where we assume that $f \in C^1$ and g is continuous. We define the total *energy* of the string by

$$E(t) = \frac{1}{2}\rho \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\tau \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx.$$

Show that the total energy of the string is conserved, in the sense that $E(t)$ is constant. Therefore,

$$E(t) = E(0) = \frac{1}{2}\rho \int_0^L g(x)^2\, dx + \frac{1}{2}\tau \int_0^L f'(x)^2\, dx.$$

Solution. We have

$$\begin{aligned}
E'(t) &= \frac{d}{dt}\left(\frac{1}{2}\rho \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\tau \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right) \\
&= \frac{1}{2}\rho \int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\tau \int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dx \\
&= \rho \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\, dx + \tau \int_0^L \frac{\partial u}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) dx \\
&= \rho \int_0^L \frac{\partial u}{\partial t}\left(c^2\frac{\partial^2 u}{\partial x^2}\right) dx + \tau \int_0^L \frac{\partial u}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) dx \\
&= \rho \int_0^L \frac{\partial u}{\partial t}\left(\frac{\tau}{\rho}\frac{\partial^2 u}{\partial x^2}\right) dx + \tau \int_0^L \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) dx \\
&= \tau\left(\int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\, dx + \int_0^L \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) dx\right).
\end{aligned}$$

The vibrating string has fixed endpoints (see page 10 of Stein-Shakarchi), which means $u(0,t) = u(L,t)$, and so, when we use integration by parts on the second term, we obtain

$$\begin{aligned}
\int_0^L \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) dx &= \frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\bigg|_0^L - \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\, dx \\
&= \frac{\partial u(L,t)}{\partial t}\frac{\partial u(L,t)}{\partial x} - \frac{\partial u(0,t)}{\partial t}\frac{\partial u(0,t)}{\partial x} - \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\, dx \\
&= \frac{\partial u(0,t)}{\partial t}\frac{\partial u(0,t)}{\partial x} - \frac{\partial u(0,t)}{\partial t}\frac{\partial u(0,t)}{\partial x} - \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\, dx \\
&= 0 - \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\, dx \\
&= -\int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\, dx.
\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}
E'(t) &= \tau \left(\int_0^L \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2}\, dx + \int_0^L \frac{\partial u}{\partial x} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) dx \right) \\
&= \tau \left(\int_0^L \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2}\, dx - \int_0^L \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2}\, dx \right) \\
&= \tau \cdot 0 \\
&= 0,
\end{aligned}$$

meaning that the total energy $E(t)$ is constant. □

3.3.12. Prove that

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}.$$

[Hint: Start with the fact that the integral of $D_N(\theta)$ equals 2π, and note that the difference $\frac{1}{\sin(\frac{\theta}{2})} - \frac{2}{\theta}$ is continuous on $[-\pi, \pi]$. Apply the Riemann-Lebesgue lemma.]

Solution. The Stein and Shakarchi textbook has defined in page 37

$$D_N(\theta) := \sum_{n=-N}^{N} e^{in\theta}$$

and established its closed form

$$D_N(\theta) = \frac{\sin((N+\frac{1}{2})\theta)}{\sin(\frac{\theta}{2})}.$$

As stated in the hint, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\sin((N+\frac{1}{2})\theta)}{\sin(\frac{\theta}{2})}\, d\theta &= \int_{-\pi}^{\pi} D_N(\theta)\, d\theta \\
&= \int_{-\pi}^{\pi} \sum_{n=-N}^{N} e^{in\theta}\, d\theta \\
&= \sum_{n=-N}^{-1} \frac{1}{in} e^{in\theta}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} e^{i(0)\theta}\, d\theta + \sum_{n=1}^{N} \frac{1}{in} e^{in\theta}\Big|_{-\pi}^{\pi} \\
&= \sum_{n=-N}^{-1} \frac{e^{in\pi} - e^{-in\pi}}{in} + \int_{-\pi}^{\pi} 1\, d\theta + \sum_{n=1}^{N} \frac{e^{in\pi} - e^{-in\pi}}{in} \\
&= \sum_{n=-N}^{-1} \frac{0}{in} + \int_{-\pi}^{\pi} 1\, d\theta + \sum_{n=1}^{N} \frac{0}{in} \\
&= \int_{-\pi}^{\pi} 1\, d\theta \\
&= \theta|_{-\pi}^{\pi} \\
&= \pi - (-\pi) \\
&= 2\pi.
\end{aligned}$$

By using for instance l'Hôpital's rule twice, we obtain

$$\begin{aligned}
\lim_{\theta\to 0}\left(\frac{1}{\sin(\frac{\theta}{2})}-\frac{2}{\theta}\right) &= \lim_{\theta\to 0}\frac{\theta-2\sin(\frac{\theta}{2})}{\theta\sin(\frac{\theta}{2})}\\
&= \lim_{\theta\to 0}\frac{\frac{d}{d\theta}(\theta-2\sin(\frac{\theta}{2}))}{\frac{d}{d\theta}(\theta\sin(\frac{\theta}{2}))}\\
&= \lim_{\theta\to 0}\frac{1-\cos(\frac{\theta}{2})}{\sin(\frac{\theta}{2})+\frac{\theta}{2}\cos(\frac{\theta}{2})}\\
&= \lim_{\theta\to 0}\frac{\frac{d}{d\theta}(1-\cos(\frac{\theta}{2}))}{\frac{d}{d\theta}(\sin(\frac{\theta}{2})+\frac{\theta}{2}\cos(\frac{\theta}{2}))}\\
&= \lim_{\theta\to 0}\frac{\frac{1}{2}\sin(\frac{\theta}{2})}{\frac{1}{2}\cos(\frac{\theta}{2})+(\frac{1}{2}\cos(\frac{\theta}{2})-\frac{\theta}{4}\sin(\frac{\theta}{2}))}\\
&= \lim_{\theta\to 0}\frac{\frac{1}{2}\sin(\frac{\theta}{2})}{\cos(\frac{\theta}{2})-\frac{\theta}{4}\sin(\frac{\theta}{2}))}\\
&= \frac{\frac{1}{2}\sin(\frac{0}{2})}{\cos(\frac{0}{2})-\frac{0}{4}\sin(\frac{0}{2}))}\\
&= \frac{0}{1-0}\\
&= 0,
\end{aligned}$$

which signifies that $\frac{1}{\sin(\frac{\theta}{2})}-\frac{2}{\theta}$ has a removable discontinuity at $\theta = 0$, and so we can regard $\frac{1}{\sin(\frac{\theta}{2})}-\frac{2}{\theta}$ as "continuous" and hence integrable on $[-\pi,\pi]$. So we can now apply the Riemann-Lebesgue Lemma in order to conclude

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}\left(\frac{1}{\sin(\frac{\theta}{2})}-\frac{2}{\theta}\right)\sin\left(\left(N+\frac{1}{2}\right)\theta\right)\,d\theta = 0.$$

By letting $x := (N+\frac{1}{2})\theta$, which implies $dx = (N+\frac{1}{2})\,d\theta$, we obtain

$$\begin{aligned}
2\pi &= \int_{-\pi}^{\pi}\frac{\sin((N+\frac{1}{2})\theta)}{\sin(\frac{\theta}{2})}\,d\theta\\
&= \int_{-\pi}^{\pi}\sin\left(\left(N+\frac{1}{2}\right)\theta\right)\left(\frac{2}{\theta}+\frac{1}{\sin(\frac{\theta}{2})}-\frac{2}{\theta}\right)d\theta\\
&= 2\int_{-\pi}^{\pi}\frac{\sin((N+\frac{1}{2})\theta)}{\theta}\,d\theta+\int_{-\pi}^{\pi}\left(\frac{1}{\sin(\frac{\theta}{2})}-\frac{2}{\theta}\right)\sin\left(\left(N+\frac{1}{2}\right)\theta\right)d\theta\\
&= 2\int_{-(N+\frac{1}{2})\pi}^{(N+\frac{1}{2})\pi}\frac{\sin(x)}{\frac{x}{N+\frac{1}{2}}}\frac{dx}{N+\frac{1}{2}}+\int_{-\pi}^{\pi}\left(\frac{1}{\sin(\frac{\theta}{2})}-\frac{2}{\theta}\right)\sin\left(\left(N+\frac{1}{2}\right)\theta\right)d\theta\\
&= 2\int_{-(N+\frac{1}{2})\pi}^{(N+\frac{1}{2})\pi}\frac{\sin(x)}{x}\,dx+\int_{-\pi}^{\pi}\left(\frac{1}{\sin(\frac{\theta}{2})}-\frac{2}{\theta}\right)\sin\left(\left(N+\frac{1}{2}\right)\theta\right)d\theta.
\end{aligned}$$

Now we send $N\to\infty$ to conclude

$$\begin{aligned}
2\pi &= \lim_{N\to\infty}2\pi\\
&= \lim_{N\to\infty}\left(2\int_{-(N+\frac{1}{2})\pi}^{(N+\frac{1}{2})\pi}\frac{\sin(x)}{x}\,dx+\int_{-\pi}^{\pi}\left(\frac{1}{\sin(\frac{\theta}{2})}-\frac{2}{\theta}\right)\sin\left(\left(N+\frac{1}{2}\right)\theta\right)d\theta\right)\\
&= 2\lim_{N\to\infty}\int_{-(N+\frac{1}{2})\pi}^{(N+\frac{1}{2})\pi}\frac{\sin(x)}{x}\,dx+\lim_{N\to\infty}\int_{-\pi}^{\pi}\left(\frac{1}{\sin(\frac{\theta}{2})}-\frac{2}{\theta}\right)\sin\left(\left(N+\frac{1}{2}\right)\theta\right)d\theta\\
&= 2\int_{-\infty}^{\infty}\frac{\sin(x)}{x}\,dx+0\\
&= 4\int_{0}^{\infty}\frac{\sin(x)}{x}\,dx,
\end{aligned}$$

or equivalently

$$\int_0^\infty\frac{\sin(x)}{x}\,dx = \frac{\pi}{2},$$

as desired. □

3.4.2. An important fact we have proved is the family $\{e^{inx}\}_{n\in\mathbb{Z}}$ is orthonormal in $\mathcal{R}$ and it is also complete, in the sense that the Fourier series of f converges in the norm. In this exercise, we consider another family possessing these same properties.

On $[-1, 1]$, define

$$L_n(x) = \frac{d^n}{dx^n}(x^2-1)^n$$

for all $n = 0, 1, 2, \ldots$. Then L_n is a polynomial of degree n, which is called the n^{th} *Legendre polynomial*.

(a) Show that if f is infinitely differentiable on $[-1, 1]$, then

$$\int_{-1}^{1} L_n(x)f(x)\,dx = (-1)^n \int_{-1}^{1} (x^2-1)^n f^{(n)}(x)\,dx.$$

In particular, show that L_n is orthogonal to x^m whenever $m < n$. Hence, $\{L_n\}_{n=0}^{\infty}$ is an orthogonal family.

Solution. I will show this by induction. You can also start at the left-hand side and integrate by parts n times to get to the right-hand side. For the base case at $n = 0$, we have

$$\begin{aligned}
\int_{-1}^{1} L_0(x)f(x)\,dx &= \int_{-1}^{1} \frac{d^0}{dx^0}(x^2-1)^0 f(x)\,dx \\
&= \int_{-1}^{1} f(x)\,dx \\
&= \int_{-1}^{1} f^{(0)}(x)\,dx \\
&= (-1)^0 \int_{-1}^{1} (x^2-1)^0 f^{(0)}(x)\,dx.
\end{aligned}$$

Now, for the inductive step, let $k = 1, 2, 3, \ldots$ assume at any $n = k$ the statement

$$\int_{-1}^{1} L_{f\,1}(x)f(x)\,dx = (-1)^k \int_{-1}^{1} (x^2-1)^k f^{(k)}(x)\,dx.$$

For $n = k+1$, we use integration by parts to obtain

$$\begin{aligned}
\int_{-1}^{1} L_{k+1}(x)f(x)\,dx &= \int_{-1}^{1} \frac{d^{k+1}}{dx^{k+1}}(x^2-1)^{k+1} f(x)\,dx \\
&= f(x)\frac{d^k}{dx^k}(x^2-1)^{k+1}\bigg|_{-1}^{1} - \int_{-1}^{1} \frac{d^k}{dx^k}(x^2-1)^{k+1} f'(x)\,dx \\
&= f(x)((k+1)!(x^2-1))\big|_{-1}^{1} - \int_{-1}^{1} \frac{d^k}{dx^k}((x^2-1)^k(x^2-1))f'(x)\,dx \\
&= (k+1)!f(x)((1^2-1)-((-1)^2-1)) \\
&\quad - \int_{-1}^{1}\left(\frac{d^k}{dx^k}(x^2-1)^k(x^2-1) + (x^2-1)\frac{d^k}{dx^k}(x^2-1)\right)f'(x)\,dx \\
&= (k+1)!f(x)(0-0) - \int_{-1}^{1}\Big(L_{f\,1}(x)(x^2-1) + (x^2-1)(0)\Big)f'(x)\,dx \\
&= -\int_{-1}^{1} L_{f\,1}(x)(x^2-1)f'(x)\,dx \\
&= -\int_{-1}^{1} L_{f\,1}(x)x^2 f'(x)\,dx + \int_{-1}^{1} L_{f\,1}(x)f'(x)\,dx \\
&= -(-1)^k \int_{-1}^{1} L_{f\,1}(x)x^2 f'(x)\,dx + \int_{-1}^{1} L_{f\,1}(x)f'(x)\,dx \\
&= (-1)^{k+1}\int_{-1}^{1}(x^2-1)^k\frac{d^k}{dx^k}(x^2f'(x))\,dx - (-1)^{k+1}\int_{-1}^{1}(x^2-1)^k f^{(k+1)}(x)\,dx \\
&= (-1)^{k+1}\int_{-1}^{1}(x^2-1)^k\left(\frac{d^k}{dx^k}(x^2)f'(x) + x^2\frac{d^k}{dx^k}f'(x)\right)dx - (-1)^{k+1}\int_{-1}^{1}(x^2-1)^k f^{(k+1)}(x)\,dx \\
&= (-1)^{k+1}\int_{-1}^{1}(x^2-1)^k(0f'(x) + x^2f^{(k+1)}(x))\,dx - (-1)^{k+1}\int_{-1}^{1}(x^2-1)^k f^{(k+1)}(x)\,dx \\
&= (-1)^{k+1}\int_{-1}^{1}(x^2-1)^k(x^2f^{(k+1)}(x) - f^{(k+1)}(x))\,dx \\
&= (-1)^{k+1}\int_{-1}^{1}(x^2-1)^k(x^2-1)f^{(k+1)}(x)\,dx \\
&= (-1)^{k+1}\int_{-1}^{1}(x^2-1)^{k+1}f^{(k+1)}(x)\,dx.
\end{aligned}$$

This completes our proof by induction. Next, we will show that L_n is orthogonal to x^m whenever $m < n$. Indeed, since we assume $m < n$, it follows that $n - m$ is positive, and so the $(n-m)^{\text{th}}$ derivative of a function—in other words, $f^{(n-m)}(x)$—makes sense. Using the identity we proved, we obtain

$$\begin{aligned}
\int_{-1}^{1} L_n(x)x^m \, dx &= -\int_{-1}^{1} (x^2-1)\frac{d^n}{dx^n}x^m \, dx \\
&= -\int_{-1}^{1} (x^2-1)\frac{d^{n-m}}{dx^{n-m}}\frac{d^m}{dx^m}x^m \, dx \\
&= -\int_{-1}^{1} (x^2-1)\frac{d^{n-m}}{dx^{n-m}}(m!) \, dx \\
&= -\int_{-1}^{1} (x^2-1)(0) \, dx \\
&= -\int_{-1}^{1} 0 \, dx \\
&= 0,
\end{aligned}$$

as desired. □

(b) Show that

$$\|L_n\|^2 = \int_{-1}^{1} |L_n(x)|^2 \, dx = \frac{(n!)^2 2^{2n+1}}{2n+1}.$$

[Hint: First note that $\|L_n\|^2 = (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n \, dx$. Write $(x^2-1)^n = (x-1)^n(x+1)^n$ and integrate by parts n times to calculate this last integral.]

Solution. Following the hint and using the identity from part (a), we have

$$\begin{aligned}
\|L_n\|^2 &= \int_{-1}^{1} |L_n(x)|^2 \, dx \\
&= \int_{-1}^{1} L_n(x)L_n(x) \, dx \\
&= \int_{-1}^{1} (x^2-1)^n L_n^{(n)}(x) \, dx \\
&= (-1)^n \int_{-1}^{1} (x^2-1)^n \frac{d^n}{dx^n} L_n(x) \, dx \\
&= (-1)^n \int_{-1}^{1} (x^2-1)^n \frac{d^n}{dx^n}\left(\frac{d^n}{dx^n}(x^2-1)^n\right) dx \\
&= (-1)^n \int_{-1}^{1} (x^2-1)^n \frac{d^{2n}}{dx^{2n}}(x^2-1)^n \, dx.
\end{aligned}$$

Next, we need to prove the claim

$$\frac{d^{2n}}{dx^{2n}}(x^2-1)^n = (2n)!$$

for all $n = 0, 1, 2, \ldots$. (What I have written below is valid but unnecessarily complicated, so please skip this part. A much shorter proof is to use the Binomial Theorem on $(x^2-1)^n$, writing

$$(x^2-1)^n = \sum_{l=0}^{n} x^{2l}(-1)^{n-l},$$

and then take the $2n^{\text{th}}$ derivative. Only the term corresponding to $l = n$ of the $2n^{\text{th}}$ derivative is nonzero, and that term is precisely $(2n)!$.) I will show this by induction. For the base case at $n = 0$, we have

$$\begin{aligned}
\frac{d^{2(0)}}{dx^{2(0)}}(x^2-1)^0 &= (x^2-1)^0 \\
&= 1 \\
&= (2(0))!
\end{aligned}$$

Now, for the inductive step, let $k = 1, 2, 3, \ldots$ assume at any $n = k$ the statement

$$\frac{d^{2k}}{dx^{2k}}(x^2-1)^k = (2k)!.$$

Then for $n = k + 1$ we use the product and chain rules for derivatives and the Binomial Theorem to obtain

$$\begin{aligned}
\frac{d^{2(k+1)}}{dx^{2(k+1)}}(x^2-1)^{k+1} &= \frac{d^{2k+2}}{dx^{2k+2}}(x^2-1)^{k+1} \\
&= \frac{d^{2k+1}}{dx^{2k+1}}\frac{d}{dx}(x^2-1)^{k+1} \\
&= \frac{d^{2k+1}}{dx^{2k+1}}(2(k+1)x(x^2-1)^k) \\
&= 2(k+1)\frac{d^{2k+1}}{dx^{2k+1}}(x(x^2-1)^k) \\
&= (2k+2)\frac{d^{2k}}{dx^{2k}}\frac{d}{dx}(x(x^2-1)^k) \\
&= (2k+2)\frac{d^{2k}}{dx^{2k}}\left(\frac{d}{dx}x(x^2-1)^k + x\frac{d}{dx}(x^2-1)^k\right) \\
&= (2k+2)\frac{d^{2k}}{dx^{2k}}(1(x^2-1)^k + 2kx^2(x^2-1)^{k-1}) \\
&= (2k+2)\left(\frac{d^{2k}}{dx^{2k}}(x^2-1)^k + 2k\frac{d^{2k}}{dx^{2k}}(x^2(x^2-1)^{k-1})\right) \\
&= (2k+2)\left((2k)! + 2k\frac{d^{2k}}{dx^{2k}}(x^2(x^2-1)^{k-1})\right) \\
&= (2k+2)\left((2k)! + 2k\frac{d^{2k}}{dx^{2k}}((x^2-1+1)(x^2-1)^{k-1})\right) \\
&= (2k+2)\left((2k)! + 2k\frac{d^{2k}}{dx^{2k}}((x^2-1)^k - (x^2-1)^{k-1})\right) \\
&= (2k+2)\left((2k)! + 2k\left(\frac{d^{2k}}{dx^{2k}}(x^2-1)^k - \frac{d^{2k}}{dx^{2k}}(x^2-1)^{k-1}\right)\right) \\
&= (2k+2)\left((2k)! + 2k\left((2k)! - \frac{d^{2k}}{dx^{2k}}(x^2-1)^{k-1}\right)\right) \\
&= (2k+2)\left((2k)! + 2k\left((2k)! - \frac{d^{2k}}{dx^{2k}}\sum_{l=0}^{k-1}(x^2)^l 1^{(k-1)-l}\right)\right) \\
&= (2k+2)\left((2k)! + 2k\left((2k)! - \sum_{l=0}^{k-1}\frac{d^{2k}}{dx^{2k}}x^{2l}\right)\right) \\
&= (2k+2)\left((2k)! + 2k\left((2k)! - \sum_{l=0}^{k-1}\frac{d^{2k-2l-1}}{dx^{2k-2l-1}}\frac{d^{2l+1}}{dx^{2l+1}}x^{2l}\right)\right) \\
&= (2k+2)\left((2k)! + 2k\left((2k)! - \sum_{l=0}^{k-1}\frac{d^{2k-2l-1}}{dx^{2k-2l-1}}(0)\right)\right) \\
&= (2k+2)((2k)! + 2k((2k)! - 0)) \\
&= (2k+2)((2k)! + 2k(2k)!) \\
&= (2k+2)(2k+1)(2k)! \\
&= (2k+2)! \\
&= (2(k+1))!.
\end{aligned}$$

This completes our proof by induction. Resume here. Putting our results together, we obtain

$$\begin{aligned}
\|L_n\|^2 &= \int_{-1}^{1} |L_n(x)|^2\, dx \\
&= (-1)^n \int_{-1}^{1} (x^2-1)^n \frac{d^{2n}}{dx^{2n}}(x^2-1)^n\, dx \\
&= (-1)^n \int_{-1}^{1} (x^2-1)^n (2n)!\, dx \\
&= (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n\, dx,
\end{aligned}$$

which establishes the first part of the textbook hint. Next, we follow the remainder of the textbook hint and apply the integration by parts n times in order to compute our integral. If we integrate by parts once, we obtain the iterative relationship

$$\begin{aligned}
\int_{-1}^{1} (x^2-1)^n \, dx &= \int_{-1}^{1} (x+1)^n (x-1)^n \, dx \\
&= \frac{(x+1)^n (x-1)^{n+1}}{n+1}\bigg|_{-1}^{1} - \frac{n}{n+1}\int_{-1}^{1} (x+1)^{n-1}(x-1)^{n+1} \, dx \\
&= \frac{(1+1)^n(1-1)^{n+1} - (-1+1)^n(-1-1)^{n+1}}{n+1} - \frac{n}{n+1}\int_{-1}^{1} (x+1)^{n-1}(x-1)^{n+1} \, dx \\
&= \frac{2^n 0^{n+1} - 0^n(-2)^{n+1}}{n+1} - \frac{n}{n+1}\int_{-1}^{1} (x+1)^{n-1}(x-1)^{n+1} \, dx \\
&= -\frac{n}{n+1}\int_{-1}^{1} (x+1)^{n-1}(x-1)^{n+1} \, dx.
\end{aligned}$$

This means that to integrate by parts n times, we need to iterate this process n times; we obtain

$$\begin{aligned}
\int_{-1}^{1} (x^2-1)^n \, dx &= -\frac{n}{n+1}\int_{-1}^{1} (x+1)^{n-1}(x-1)^{n+1} \, dx \\
&= \left(-\frac{n}{n+1}\right)\left(-\frac{n-1}{n+2}\int_{-1}^{1} (x+1)^{n-2}(x-1)^{n+2} \, dx\right) \\
&= \left(-\frac{n}{n+1}\right)\left(-\frac{n-1}{n+2}\right)\left(-\frac{n-2}{n+3}\int_{-1}^{1} (x+1)^{n-2}(x-1)^{n+2} \, dx\right) \\
&\vdots \text{ We iterated 2 times so far. So we continue this process } n-2 \text{ more times.} \\
&= \left(-\frac{n}{n+1}\right)\left(-\frac{n-1}{n+2}\right)\left(-\frac{n-2}{n+3}\right)\cdots\left(-\frac{n-(n-1)}{n+n}\int_{-1}^{1} (x+1)^{n-n}(x-1)^{n+n} \, dx\right) \\
&= \frac{(-1)^n n!}{(2n)\cdots(n+3)(n+2)(n+1)}\int_{-1}^{1} (x-1)^{2n} \, dx \\
&= \frac{(-1)^n (n!)^2}{(2n)\cdots(n+3)(n+2)(n+1)n!}\frac{(x-1)^{2n+1}}{2n+1}\bigg|_{-1}^{1} \\
&= \frac{(-1)^n (n!)^2}{(2n)!}\frac{(1-1)^{2n+1} - (-1-1)^{2n+1}}{2n+1} \\
&= \frac{(-1)^n (n!)^2}{(2n)!}\frac{0^{2n+1} - (-2)^{2n+1}}{2n+1} \\
&= \frac{(-1)^n (n!)^2}{(2n)!}\frac{-(-2)(-2)^{2n}}{2n+1} \\
&= \frac{(-1)^n (n!)^2}{(2n)!}\frac{2(2)^{2n}}{2n+1} \\
&= \frac{(-1)^n (n!)^2 2^{2n+1}}{(2n)!(2n+1)}.
\end{aligned}$$

Finally, we conclude

$$\begin{aligned}
\|L_n\|^2 &= (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n (2n)! \, dx \\
&= (-1)^n (2n)! \frac{(-1)^n (n!)^2 2^{2n+1}}{(2n)!(2n+1)} \\
&= \frac{(-1)^{2n} (n!)^2 2^{2n+1}}{2n+1} \\
&= \frac{(n!)^2 2^{2n+1}}{2n+1},
\end{aligned}$$

as desired. □

(c) Prove that any polynomial of degree n that is orthogonal to $1, x, x^2, \ldots, x^{n-1}$ is a constant multiple of L_n.

Solution. I do not know the answer to this one. This "proof" is probably invalid. Let p_n be a polynomial of degree n on $[-1, 1]$ that is orthogonal to $1, x, x^2, \ldots, x^{n-1}$. Then we must have

$$\int_{-1}^{1} p(x)x^k \, dx = 0.$$

for any $k = 0, 1, 2, \ldots, n-1$. Suppose by contradiction that such a polyonimal p_n is not a constant multiple of L_n. Then we would have

$$p_n(x) \neq \lambda L_n(x)$$

for all $\lambda \in \mathbb{R}$ and for all $x \in [-1, 1]$. But then by part (a) we would have

$$\begin{aligned}\int_{-1}^{1} p(x)x^k \, dx &\neq \int_{-1}^{1} \lambda L_n(x)x^k \, dx \\ &= \lambda \int_{-1}^{1} L_n(x)x^k \, dx \\ &= \lambda \cdot 0 \\ &= 0,\end{aligned}$$

but this contradicts our assumption that p_n is orthogonal to x^k for any $k = 0, 1, 2, \ldots, n-1$. □

(d) Let $\mathcal{L}_n = \frac{L_n}{\|L_n\|}$, which are the normalized Legendre polynomials. Prove that $\{\mathcal{L}_n\}$ is the family obtained by applying the "Gram-Schmidt process" to $\{1, x, x^2, \ldots, x^n, \ldots\}$, and conclude that every Riemann integrable function f on $[-1, 1]$ has a *Legendre expansion*

$$\sum_{n=0}^{\infty} \langle f, \mathcal{L}_n \rangle \mathcal{L}_n$$

which converges to f in the mean-square sense.

Solution. Since the Gram-Schmit process was not defined anywhere in the textbook, I will follow the process outlined on the corresponding Wikipedia article. To prove that $\{\mathcal{L}_n\}$ is the family means, in this context, to prove that $\{\mathcal{L}_n\}$ is orthonormal. Since we have

$$\begin{aligned}\|\mathcal{L}_n\| &= \left\| \frac{L_n}{\|L_n\|} \right\| \\ &= \frac{\|L_n\|}{\|L_n\|} \\ &= 1,\end{aligned}$$

we already see that $\mathcal{L}_n$ is normal. To show that $\{\mathcal{L}_n\}$ is orthonormal, we apply the "Gram-Schmidt process" to $\{1, x, x^2, \ldots, x^n, \ldots\}$ to construct an orthonormal basis $\{w_0(x), w_1(x), w_2(x), \ldots, w_n(x), \ldots\}$ given by

$$w_0(x) := 1$$

and

$$\begin{aligned}w_1(x) &:= x - \frac{\langle x, w_0(x) \rangle}{\|w_0(x)\|^2} w_0(x) \\ &= x - \frac{\int_{-1}^{1} x \cdot 1 \, dx}{\int_{-1}^{1} 1 \, dx} 1 \\ &= x - \frac{0}{2} 1 \\ &= x\end{aligned}$$

and

$$\begin{aligned}w_2(x) &:= x^2 - \frac{\langle x^2, w_0(x) \rangle}{\|w_0(x)\|^2} w_0(x) - \frac{\langle x^2, w_1(x) \rangle}{\|w_1(x)\|^2} w_1(x) \\ &= x^2 - \frac{\int_{-1}^{1} x^2 \cdot 1 \, dx}{\int_{-1}^{1} 1^2 \, dx} 1 - \frac{\int_{-1}^{1} x^2 x \, dx}{\int_{-1}^{1} x^2 \, dx} x \\ &= x^2 - \frac{\frac{2}{3}}{2} 1 - \frac{0}{\frac{2}{3}} x \\ &= x^2 - \frac{1}{3}\end{aligned}$$

and

$$\begin{aligned}
w_3(x) &:= x^3 - \frac{\langle x^3, w_0(x)\rangle}{\|w_0(x)\|^2} w_0(x) - \frac{\langle x^3, w_1(x)\rangle}{\|w_1(x)\|^2} w_1(x) - \frac{\langle x^3, w_2(x)\rangle}{\|w_2(x)\|^2} w_2(x) \\
&= x^3 - \frac{\int_{-1}^{1} x^3 \cdot 1\, dx}{\int_{-1}^{1} 1^2\, dx} 1 - \frac{\int_{-1}^{1} x^3 x\, dx}{\int_{-1}^{1} x^2\, dx} x - \frac{\int_{-1}^{1} x^3 (x^2 - \frac{1}{3})\, dx}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2\, dx} \left(x^2 - \frac{1}{3}\right) \\
&= x^3 - \frac{0}{\frac{2}{3}} 1 - \frac{\frac{2}{5}}{\frac{2}{3}} x - \frac{0}{\frac{8}{45}} \left(x^2 - \frac{1}{3}\right) \\
&= x^3 - \frac{3}{5} x
\end{aligned}$$

and so on. We also have

$$\begin{aligned}
L_0(x) &= \frac{d^0}{dx^0} (x^2 - 1)^0 \\
&= (x^2 - 1)^0 \\
&= 1 \\
&= w_0(x)
\end{aligned}$$

and

$$\begin{aligned}
L_1(x) &= \frac{d^1}{dx^1} (x^2 - 1)^1 \\
&= \frac{d}{dx} (x^2 - 1) \\
&= 2x \\
&= 2w_1(x)
\end{aligned}$$

and

$$\begin{aligned}
L_2(x) &= \frac{d^2}{dx^2} (x^2 - 1)^2 \\
&= 12x^2 - 4 \\
&= 12 \left(x^2 - \frac{1}{3}\right) \\
&= 12 w_2(x)
\end{aligned}$$

and

$$\begin{aligned}
L_3(x) &= \frac{d^3}{dx^3} (x^2 - 1)^3 \\
&= 120x^3 - 72x \\
&= 120 \left(x^3 - \frac{3}{5} x\right) \\
&= 120 w_3(x),
\end{aligned}$$

and so on. We can continue these processes infinitely many times—computing in general the n^{th} terms $w_n(x)$ and $L_n(x)$—to see that each term $L_n(x)$ is a scalar multiple of $w_n(x)$. Therefore, since $\{w_0(x), w_1(x), w_2(x), \ldots, w_n(x), \ldots\}$ is an orthonormal basis, it follows that $\{L_0(x), L_1(x), L_2(x), \ldots, L_n(x), \ldots\}$ is an orthogonal basis, from which we can immediately conclude that $\{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n, \ldots\}$ is an orthonormal basis.
Next, we will establish the Legendre expansion

$$f = \sum_{n=0}^{\infty} \langle f, \mathcal{L}_n \rangle \mathcal{L}_n$$

by showing that the sum

$$S_N(f) := \sum_{n=0}^{N} \langle f, \mathcal{L}_n \rangle \mathcal{L}_n$$

converges to f in the mean-square sense. To this end, let $\epsilon > 0$ be given. By the Weierstrass Approximation Theorem, there exists a polynomial $p(x)$ of degree n defined on $[-1, 1]$ that satisfies $\|f - p\| < \epsilon$. By part (c), any polynomial is a constant multiple of L_n, and in turn a constant multiple of $\mathcal{L}_n$. In particular, we have

$$\|f - S_N(f)\| < \epsilon,$$

which means $S_N(f)$ converges to f in the mean-square sense, as desired. □

3.4.3. Let α be a complex number not equal to an integer.

(a) Calculate the Fourier series of the 2π-periodic functiondefined on $[-\pi, \pi]$ by $f(x) = \cos(\alpha x)$.

Solution. For all $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, we have the Fourier cosine coefficient

$$\begin{aligned}
a_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(\alpha x) \cos(nx)\, dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}(\cos(\alpha x + nx) + \cos(\alpha x - nx))\, dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((n+\alpha)x) + \cos((\alpha - n)x)\, dx \\
&= \frac{1}{2\pi} \left(\frac{\sin((n+\alpha)x)}{n+\alpha} + \frac{\sin((\alpha-n)x)}{\alpha - n} \right) \Bigg|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left(\frac{\sin(\alpha x + nx)}{n+\alpha} + \frac{\sin(\alpha x - nx)}{\alpha - n} \right) \Bigg|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left(\frac{\sin(\alpha x)\cos(nx) + \sin(nx)\cos(\alpha x)}{n+\alpha} + \frac{\sin(\alpha x)\cos(nx) - \sin(nx)\cos(\alpha x)}{\alpha - n} \right) \Bigg|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left(\frac{\sin(\alpha\pi)\cos(n\pi) + \sin(n\pi)\cos(\alpha\pi)}{n+\alpha} - \frac{\sin(\alpha(-\pi))\cos(n(-\pi)) + \sin(n(-\pi))\cos(\alpha(-\pi))}{n+\alpha} \right. \\
&\qquad \left. + \frac{\sin(\alpha\pi)\cos(n\pi) - \sin(n\pi)\cos(\alpha\pi)}{\alpha - n} - \frac{\sin(\alpha(-\pi))\cos(n(-\pi)) - \sin(n(-\pi))\cos(\alpha(-\pi))}{\alpha - n} \right) \\
&= \frac{1}{2\pi} \left(\frac{\sin(\alpha\pi)(-1)^n + 0\cos(\alpha\pi)}{n+\alpha} - \frac{-\sin(\alpha\pi)(-1)^n + 0\cos(\alpha\pi)}{n+\alpha} \right. \\
&\qquad \left. + \frac{\sin(\alpha\pi)(-1)^n - 0\cos(\alpha\pi)}{\alpha - n} - \frac{-\sin(\alpha\pi)(-1)^n - 0\cos(\alpha\pi)}{\alpha - n} \right) \\
&= \frac{(-1)^n}{2\pi} \left(\frac{2\sin(\alpha\pi)}{n+\alpha} + \frac{2\sin(\alpha\pi)}{\alpha - n} \right) \\
&= \frac{(-1)^n}{2\pi} \frac{4\alpha \sin(\alpha\pi)}{\alpha^2 - n^2} \\
&= \frac{(-1)^n 2\alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)}
\end{aligned}$$

and the Fourier sine coefficient

$$
\begin{aligned}
b_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\alpha x)\sin(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}(\sin(\alpha x + nx) + \sin(\alpha x - nx))\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin((n+\alpha)x) + \sin((\alpha - n)x)\,dx \\
&= \frac{1}{2\pi}\left(-\frac{\cos((n+\alpha)x)}{n+\alpha} - \frac{\cos((\alpha-n)x)}{\alpha-n}\right)\Bigg|_{-\pi}^{\pi} \\
&= -\frac{1}{2\pi}\left(\frac{\cos(\alpha x + nx)}{n+\alpha} + \frac{\cos(\alpha x - nx)}{\alpha - n}\right)\Bigg|_{-\pi}^{\pi} \\
&= -\frac{1}{2\pi}\left(\frac{\cos(\alpha x)\cos(nx) - \sin(nx)\sin(\alpha x)}{n+\alpha} + \frac{\cos(\alpha x)\cos(nx) + \sin(nx)\sin(\alpha x)}{\alpha - n}\right)\Bigg|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\Bigg(\frac{\cos(\alpha\pi)\cos(n\pi) - \sin(n\pi)\sin(\alpha\pi)}{n+\alpha} - \frac{\cos(\alpha(-\pi))\cos(n(-\pi)) - \sin(n(-\pi))\sin(\alpha(-\pi))}{n+\alpha} \\
&\qquad + \frac{\cos(\alpha\pi)\cos(n\pi) + \sin(n\pi)\sin(\alpha\pi)}{\alpha - n} - \frac{\cos(\alpha(-\pi))\cos(n(-\pi)) + \sin(n(-\pi))\sin(\alpha(-\pi))}{\alpha - n}\Bigg) \\
&= \frac{1}{2\pi}\Bigg(\frac{\cos(\alpha\pi)(-1)^n + 0\sin(\alpha\pi)}{n+\alpha} - \frac{\cos(\alpha\pi)(-1)^n + 0\sin(\alpha\pi)}{n+\alpha} \\
&\qquad + \frac{\cos(\alpha\pi)(-1)^n - 0\sin(\alpha\pi)}{\alpha - n} - \frac{\cos(\alpha\pi)(-1)^n - 0\sin(\alpha\pi)}{\alpha - n}\Bigg) \\
&= \frac{1}{2\pi}\left(\frac{0}{n+\alpha} + \frac{0}{\alpha - n}\right) \\
&= 0,
\end{aligned}
$$

which implies the complex Fourier coefficient

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos(nx) - i\sin(x))\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx - i\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin(x)\,dx \\
&= \frac{a_n}{2} - i\frac{b_n}{2} \\
&= \frac{1}{2}\frac{(-1)^n 2\alpha\sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} - i\frac{0}{2} \\
&= \frac{(-1)^n\alpha\sin(\alpha\pi)}{\pi(\alpha^2 - n^2)}
\end{aligned}
$$

for all integers n. The Fourier series in sine-cosine form is

$$\begin{aligned}
f(x) &\sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} e^{inx} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} (\cos(nx) + i\sin(nx)) \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \cos(nx) + i \sum_{n=-\infty}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \sin(nx) \\
&= \sum_{n=-\infty}^{-1} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \cos(nx) + \frac{(-1)^0 \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - 0^2)} \cos(0x) + \sum_{n=1}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \cos(nx) \\
&\quad + i \left(\sum_{n=-\infty}^{-1} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \sin(nx) + \frac{(-1)^0 \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - 0^2)} \sin(0x) + \sum_{n=1}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \sin(nx) \right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{-n} \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - (-n)^2)} \cos(-nx) + \frac{\sin(\alpha\pi)}{\alpha\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \cos(nx) \\
&\quad + i \left(\sum_{n=1}^{\infty} \frac{(-1)^{-n} \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - (-n)^2)} \sin(-nx) + 0 + \sum_{n=1}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \sin(nx) \right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \cos(nx) + \frac{\sin(\alpha\pi)}{\alpha\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \cos(nx) \\
&\quad + i \left(-\sum_{n=1}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \sin(nx) + \sum_{n=1}^{\infty} \frac{(-1)^n \alpha \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \sin(nx) \right) \\
&= \frac{\sin(\alpha\pi)}{\alpha\pi} + 2\alpha \sum_{n=1}^{\infty} \frac{(-1)^n \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \cos(nx) + i0 \\
&= \boxed{\frac{\sin(\alpha\pi)}{\alpha\pi} + 2\alpha \sum_{n=1}^{\infty} \frac{(-1)^n \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \cos(nx)}
\end{aligned}$$

for all $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. □

(b) Prove the following formulas due to Euler:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha\tan(\alpha\pi)}.$$

For all $u \in \mathbb{C} \setminus n\mathbb{Z}$,

$$\cot(u) = \frac{1}{u} + 2\sum_{n=1}^{\infty} \frac{u}{u^2 - n^2\pi^2}.$$

Solution. At $x = \pi$, we have $f(\pi) = \cos(\alpha\pi)$, and the Fourier series becomes

$$\begin{aligned}
f(\pi) &= \frac{\sin(\alpha\pi)}{\alpha\pi} + 2\alpha \sum_{n=1}^{\infty} \frac{(-1)^n \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \cos(n\pi) \\
&= \frac{\sin(\alpha\pi)}{\alpha\pi} + 2\alpha \sum_{n=1}^{\infty} \frac{(-1)^n \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} (-1)^n \\
&= \frac{\sin(\alpha\pi)}{\alpha\pi} + 2\alpha \sum_{n=1}^{\infty} \frac{(-1)^{2n} \sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \\
&= \frac{\sin(\alpha\pi)}{\alpha\pi} + 2\alpha \sum_{n=1}^{\infty} \frac{\sin(\alpha\pi)}{\pi(\alpha^2 - n^2)} \\
&= \frac{\sin(\alpha\pi)}{\alpha\pi} - \frac{2\alpha\sin(\alpha\pi)}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2}.
\end{aligned}$$

So we conclude

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2-\alpha^2} &= \frac{\pi}{2\alpha\sin(\alpha\pi)}\left(\frac{\sin(\alpha\pi)}{\alpha\pi}-f(\pi)\right)\\ &= \frac{\pi}{2\alpha\sin(\alpha\pi)}\left(\frac{\sin(\alpha\pi)}{\alpha\pi}-\cos(\alpha\pi)\right)\\ &= \frac{1}{2\alpha^2}-\frac{\pi\cos(\alpha\pi)}{2\alpha\sin(\alpha\pi)}\\ &= \frac{1}{2\alpha^2}-\frac{\pi}{2\alpha\tan(\alpha\pi)}\end{aligned}$$

for all $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, which is the first identity. Furthermore, if we substitute $u = \alpha\pi$, then the first identity becomes

$$\sum_{n=1}^{\infty}\frac{1}{n^2-(\frac{u}{\pi})^2} = \frac{1}{2(\frac{u}{\pi})^2}-\frac{\pi}{2\frac{u}{\pi}\tan(u)},$$

which can be rewritten as

$$\sum_{n=1}^{\infty}\frac{\pi^2}{n^2\pi^2-u^2} = \frac{\pi^2}{2u^2}-\frac{\pi^2}{2u\tan(u)},$$

from which we candivide both sides by π^2 to obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2-u^2} = \frac{1}{2u^2}-\frac{1}{2u\tan(u)}.$$

So we conclude

$$\begin{aligned}\cot(u) &= \frac{1}{\tan(u)}\\ &= \frac{2u}{2u\tan(u)}\\ &= 2u\left(\frac{1}{2u^2}-\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2-u^2}\right)\\ &= 2u\left(\frac{1}{2u^2}+\sum_{n=1}^{\infty}\frac{1}{u^2-n^2\pi^2}\right)\\ &= \frac{1}{u}+2\sum_{n=1}^{\infty}\frac{u}{u^2-n^2\pi^2}\end{aligned}$$

for all $u \in \mathbb{C}\setminus n\mathbb{Z}$, which is the second identity. □

(c) Show that for all $\alpha \in \mathbb{C}\setminus\mathbb{Z}$ we have

$$\frac{\alpha\pi}{\sin(\alpha\pi)} = 1+2\alpha^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-\alpha^2}.$$

Solution. At $x = 0$, we have $f(\pi) = \cos(\alpha(0)) = 1$, and the Fourier series becomes

$$\begin{aligned}f(\pi) &= \frac{\sin(\alpha\pi)}{\alpha\pi}+2\alpha\sum_{n=1}^{\infty}\frac{(-1)^n\sin(\alpha\pi)}{\pi(\alpha^2-n^2)}\cos(n(0))\\ &= \frac{\sin(\alpha\pi)}{\alpha\pi}+2\alpha\sum_{n=1}^{\infty}\frac{(-1)^n\sin(\alpha\pi)}{\pi(\alpha^2-n^2)}\\ &= \frac{\sin(\alpha\pi)}{\alpha\pi}+\frac{2\alpha\sin(\alpha\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)}{\alpha^2-n^2}\\ &= \frac{\sin(\alpha\pi)}{\alpha\pi}+\frac{2\alpha\sin(\alpha\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-\alpha^2}.\end{aligned}$$

So we conclude

$$
\begin{aligned}
\frac{\alpha\pi}{\sin(\alpha\pi)} &= \frac{\alpha\pi}{\sin(\alpha\pi)} \cdot 1 \\
&= \frac{\alpha\pi}{\sin(\alpha\pi)} f(\pi) \\
&= \frac{\alpha\pi}{\sin(\alpha\pi)} \left(\frac{\sin(\alpha\pi)}{\alpha\pi} + \frac{2\alpha\sin(\alpha\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2-\alpha^2} \right) \\
&= 1 + 2\alpha^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2-\alpha^2}
\end{aligned}
$$

for all $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. □

(d) For all $0 < \alpha < 1$, show that

$$\int_0^\infty \frac{t^{\alpha-1}}{t+1}\, dt = \frac{\pi}{\sin(\alpha\pi)}.$$

[Hint: Split the integral as $\int_0^1 + \int_1^\infty$ and change variables $t = \frac{1}{u}$ in the second integral. Now both integrals are of the form

$$\int_0^1 \frac{t^{\gamma-1}}{1+t}\, dt$$

for all $0 < \gamma < 1$, which one can show is equal to $\sum_{n=0}^\infty \frac{(-1)^n}{n+\gamma}$. Use part (c) to conclude the proof.]

Solution. Following the hint, we split the integral by writing

$$\int_0^\infty \frac{t^{\alpha-1}}{t+1}\, dt = \int_0^1 \frac{t^{\alpha-1}}{t+1}\, dt + \int_1^\infty \frac{t^{\alpha-1}}{t+1}\, dt.$$

Employing the substitution $t = \frac{1}{u}$, which implies $dt = -\frac{1}{u^2} du$, our second integral becomes

$$
\begin{aligned}
\int_1^\infty \frac{t^{\alpha-1}}{t+1}\, dt &= \int_1^0 \frac{(\frac{1}{u})^{\alpha-1}}{\frac{1}{u}+1} \left(-\frac{1}{u^2}\, du \right) \\
&= \int_0^1 \frac{(\frac{1}{u})^{\alpha-1}}{\frac{1}{u}+1} \frac{1}{u^2}\, du \\
&= \int_0^1 \frac{1}{u^\alpha(1+u)}\, du \\
&= \int_0^1 \frac{u^{(1-\alpha)-1}}{u+1}\, du \\
&= \int_0^1 \frac{t^{(1-\alpha)-1}}{t+1}\, dt.
\end{aligned}
$$

Therefore, we can express our integral as

$$
\begin{aligned}
\int_0^\infty \frac{t^{\alpha-1}}{t+1}\, dt &= \int_0^1 \frac{t^{\alpha-1}}{t+1}\, dt + \int_1^\infty \frac{t^{\alpha-1}}{t+1}\, dt \\
&= \int_0^1 \frac{t^{\alpha-1}}{t+1}\, dt + \int_0^1 \frac{t^{(1-\alpha)-1}}{t+1}\, dt,
\end{aligned}
$$

meaning that the requested integral is expressed as the sum of two integrals of the form $\int_0^1 \frac{t^{\gamma-1}}{t+1}\, dt$ for any $0 < \gamma < 1$. Next, we will continue following the given hint by establishing

$$\int_0^1 \frac{t^{\gamma-1}}{t+1}\, dt = \sum_{n=0}^\infty \frac{(-1)^n}{n+\gamma}$$

for any $0 < \gamma < 1$. Fix $0 < s < 1$ and observe that for all $0 \le t \le s$, we invoke the geometric sum formula to obtain

$$
\begin{aligned}
\sum_{n=0}^\infty (-1)^n t^n &= \sum_{n=0}^\infty (-t)^n \\
&= \frac{1}{1-(-t)} \\
&= \frac{1}{t+1},
\end{aligned}
$$

which implies

$$\begin{aligned}\int_0^s \frac{t^{\gamma-1}}{t+1}\,dt &= \int_0^s t^{\gamma-1}\sum_{n=0}^{\infty}(-1)^n t^n\,dt\\ &= \sum_{n=0}^{\infty}(-1)^n \int_0^s t^{n+\gamma-1}\,dt\\ &= \sum_{n=0}^{\infty}(-1)^n \left.\frac{t^{n+\gamma}}{n+\gamma}\right|_0^s\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{s^{n+\gamma}-0^{n+\gamma}}{n+\gamma}\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{s^{n+\gamma}}{n+\gamma}.\end{aligned}$$

Furthermore, this series is Abel summable, and so by Abel's Theorem we obtain

$$\begin{aligned}\int_0^1 \frac{t^{\gamma-1}}{t+1}\,dt &= \lim_{s\to 1^-}\int_0^s \frac{t^{\gamma-1}}{t+1}\,dt\\ &= \lim_{s\to 1^-}\sum_{n=0}^{\infty}(-1)^n \frac{s^{n+\gamma}}{n+\gamma}\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{1^{n+\gamma}}{n+\gamma}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+\gamma}.\end{aligned}$$

Finally, we conclude

$$\begin{aligned}\int_0^\infty \frac{t^{\alpha-1}}{t+1}\,dt &= \int_0^1 \frac{t^{\alpha-1}}{t+1}\,dt + \int_0^1 \frac{t^{(1-\alpha)-1}}{t+1}\,dt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+\alpha} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+(1-\alpha)}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+\alpha} + \sum_{n=0}^{\infty}\frac{(-1)^{(n+1)-1}}{(n+1)-\alpha}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+\alpha} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n-\alpha}\\ &= \left(\frac{(-1)^0}{0+\alpha} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n+\alpha}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n-\alpha}\\ &= \frac{1}{\alpha} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)}{n+\alpha} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n-\alpha}\\ &= \frac{1}{\alpha} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+\alpha} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n-\alpha}\\ &= \frac{1}{\alpha} + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n-\alpha} - \frac{1}{n+\alpha}\right)\\ &= \frac{1}{\alpha} + 2\alpha\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-\alpha^2}\\ &= \frac{1}{\alpha}\left(1 + 2\alpha^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-\alpha^2}\right)\\ &= \frac{1}{\alpha}\frac{\alpha\pi}{\sin(\alpha\pi)}\\ &= \frac{\pi}{\sin(\alpha\pi)},\end{aligned}$$

where we invoked part (c) for our second-to-last last equality above, as given by the final part of the textbook hint. □