MATH 147 discussion
University of California, Riverside

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Solutions to assigned homework problems from *Fourier Analysis: An Introduction* by Elias Stein and Rami Sakarchi

Homework 3

- Sect. 2.6, pp. 58-65: 8, 10, 12*, 15*, 16*
- Sect. 5.5, pp. 161-169: 1, 2, 6, 11*, 15*, 16*, 21
- Sect. 6.6: pp. 207-212: 6*, 7, 8*, 10*, 11*, 12*

  *=optional

Note: I have not included any solutions to all optional exercises for this problem set.

2.6.8. Verify that $\frac{1}{2i}\sum_{n\neq 0}\frac{e^{inx}}{n}$ is the Fourier series of the 2π-periodic **sawtooth** function defined by $f(0)=0$ and

$$f(x)=\begin{cases}-\frac{\pi}{2}-\frac{x}{2} & \text{if } -\pi<x<0,\\ \frac{\pi}{2}-\frac{x}{2} & \text{if } 0<x<\pi.\end{cases}$$

Note that this function is not continuous. Show that nevertheless, the series converges for every x (by which we mean, as usual, that the symmetric partial sums of the series converge). In particular, the value of the series at the origin, namely 0, is the average of the values of $f(x)$ as x approaches the origin from the left and the right.

[Hint: Use Dirichlet's test for convergence of a series $\sum_{n=-\infty}^{\infty}a_nb_n$.]

Solution. □

2.6.10. Suppose f is a periodic function of period 2π which belongs to the class C^k. Show that

$$\hat{f}(n)=O\left(\frac{1}{n^k}\right)$$

as $|n|\to\infty$. This notation means that there exists a constant C such $|\hat{f}(n)|\leq\frac{C}{|n|^k}$. We could also write this as $|n|^k\hat{f}(n)=O(1)$, where $O(1)$ means bounded.

[Hint: Integrate by parts.]

Solution. □

5.5.1. Corollary 2.3 in Chapter 2 leads to the following simplified version of the Fourier inversion formula. Suppose f is a continuous function supported on an interval $[M,M]$, whose Fourier transform $\hat{f}$ is of moderate decrease.

(a) Fix L with $\frac{L}{2}>M$, and show that

$$f(x)=\sum_{n=-\infty}^{\infty}a_n(L)e^{\frac{2\pi inx}{L}},$$

where

$$a_n(L)=\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}f(x)e^{-\frac{2\pi inx}{L}}\,dx=\frac{1}{L}\hat{f}\left(\frac{n}{L}\right).$$

Alternatively, we may write $f(x)=\delta\sum_{n=-\infty}^{\infty}\hat{f}(n\delta)e^{2\pi in\delta x}$ with $\delta=\frac{1}{L}$.

[Hint: Note that the Fourier series of f on $[-\frac{L}{2},\frac{L}{2}]$ converges absolutely.]

Solution. Since $\hat{f}$ is of moderate decrease, there exists $A>0$ that satisfies

$$|\hat{f}(n)|\leq\frac{A}{1+n^2}.$$

So we have

$$\begin{aligned}|a_n(L)|&=\frac{1}{L}\left|\hat{f}\left(\frac{n}{L}\right)\right|\\&\leq\frac{1}{L}\frac{A}{1+(\frac{n}{L})^2}\\&=\frac{1}{L}\frac{AL^2}{L^2+n^2}\\&=\frac{AL}{L^2+n^2}\\&\leq\frac{AL}{n^2},\end{aligned}$$

which implies

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |\hat{f}(n)| &\leq \sum_{n=-\infty}^{\infty} \frac{AL}{n^2} \\ &= AL \sum_{n=-\infty}^{\infty} \frac{1}{n^2} \\ &< \infty,\end{aligned}$$

meaning that the Fourier series of f is absolutely convergent. Now consider the N^{th} partial sum

$$S_N(f) := \sum_{n=-N}^{N} a_n(L) e^{\frac{2\pi i n x}{L}}.$$

We conclude, using Corollary 2.3 in Chapter 2, that $\{S_N\}$ converges uniformly to f, and so we have

$$\begin{aligned}f(x) &= \lim_{n\to\infty} S_N(f) \\ &= \lim_{n\to\infty} \sum_{n=-N}^{N} a_n(L) e^{\frac{2\pi i n x}{L}} |\hat{f}(n)| \\ &= \sum_{n=-\infty}^{\infty} a_n(L) e^{\frac{2\pi i n x}{L}} |\hat{f}(n)|,\end{aligned}$$

as desired. □

(b) Prove that if F is continuous and of moderate decrease, then

$$\int_{-\infty}^{\infty} F(\xi)\, d\xi = \lim_{\substack{\delta\to 0 \\ \delta>0}} \delta \sum_{n=-\infty}^{\infty} F(\delta n).$$

[Hint: First approximate the integral by $\int_{-N}^{N} F$ and the sum by $\delta \sum_{n=-\infty}^{\infty}$. Then approximate the second integral by Riemann sums.]

Solution. Let $\epsilon > 0$ be given. Since F is of moderate decrease, it is continuous and hence integrable the bounded interval $[-N, N]$, where $N > 0$ is large, and since $\delta \sum_{|\delta n|\leq N} F(\delta n)$ is sufficiently close to a Riemann sum, there exists $\delta_1 > 0$ such that, for all $0 < \delta < \delta_1$, we have

$$\left| \int_{-N}^{N} F(\xi)\, d\xi - \delta \sum_{|\delta n|\leq N} F(\delta n) \right| < \epsilon$$

Since F is of moderate decrease, there exists $A > 0$ that satisfies

$$|F(x)| \leq \frac{A}{1+|x|^2},$$

and since N is large (such as $N > \frac{2A}{\epsilon}$), we have

$$\begin{aligned}\left|\int_{|x|\geq N} F(\xi)\, d\xi\right| &\leq \int_{|x|\geq N} |F(\xi)|\, d\xi \\ &\leq \int_{|x|\geq N} \frac{A}{1+|x|^2}\, d\xi \\ &= A \int_{|x|\geq N} \frac{1}{1+|x|^2}\, d\xi \\ &= 2A \int_{N}^{\infty} \frac{1}{1+|x|^2}\, d\xi \\ &< 2A \int_{N}^{\infty} \frac{1}{|x|^2}\, d\xi \\ &= -2A \frac{1}{x}\Big|_{N}^{\infty} \\ &= -2A\left(0 - \frac{1}{N}\right) \\ &= \frac{2A}{N} \\ &< \epsilon.\end{aligned}$$

Finally, there exists a small $\delta_2 > 0$ such that, for all $0 < \delta < \delta_2$, we have

$$\begin{aligned}
\left|\delta \sum_{|\delta n|\geq N} F(\delta n)\right| &\leq \delta \sum_{|\delta n|\geq N} |F(\delta n)| \\
&\leq \delta \sum_{|\delta n|\geq N} \frac{A}{1+|\delta n|^2} \\
&= 2A\delta \sum_{\delta n\geq N} \frac{1}{1+\delta^2 n^2} \\
&< 2A\delta \sum_{\delta n\geq N} \frac{1}{\delta^2 n^2} \\
&\leq 2A\delta \int_N^\infty \frac{1}{x^2}\, dx \\
&= -2A\delta \frac{1}{x}\bigg|_N^\infty \\
&= -2A\delta\left(0 - \frac{1}{N}\right) \\
&= \frac{2A\delta}{N} \\
&< \frac{2A\delta_2}{N} \\
&\leq \frac{2A}{N} \\
&< \epsilon.
\end{aligned}$$

Finally, if we choose $\delta := \min\{\delta_1, \delta_2\}$, we conclude

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} F(\xi)\, d\xi - \delta \sum_{n=-\infty}^{\infty} F(\delta n)\right| &= \left|\int_{-N}^{N} F(\xi)\, d\xi - \delta \sum_{|\delta n|\leq N} F(\delta n) + \int_{|\xi|\geq N} F(\xi)\, d\xi + \delta \sum_{|\delta n|>N} F(\delta n)\right| \\
&\leq \left|\int_{-N}^{N} F(\xi)\, d\xi - \delta \sum_{|\delta n|\leq N} F(\delta n)\right| + \left|\int_{|\xi|\geq N} F(\xi)\, d\xi\right| + \left|\delta \sum_{|\delta n|>N} F(\delta n)\right| \\
&< \epsilon + \epsilon + \epsilon \\
&= 3\epsilon,
\end{aligned}$$

which means

$$\int_{-\infty}^{\infty} F(\xi)\, d\xi = \lim_{\substack{\delta\to 0\\ \delta>0}} \delta \sum_{n=-\infty}^{\infty} F(\delta n),$$

as desired. □

(c) Conclude that $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi$.

Solution. Set $F(\xi) := \hat{f}(\xi)e^{2\pi i\xi x}$ and choose $\delta = \frac{1}{L}$. Then by part (b) we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i\xi x}\, d\xi &= \int_{-\infty}^{\infty} F(\xi)\, d\xi \\
&= \lim_{\substack{\delta\to 0\\ \delta>0}} \delta \sum_{n=-\infty}^{\infty} F(\delta n) \\
&= \lim_{\substack{\delta\to 0\\ \delta>0}} \delta \sum_{n=-\infty}^{\infty} \hat{f}(\delta n)e^{2\pi i\delta n x} \\
&= \lim_{L\to\infty} \frac{1}{L} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{L}\right) e^{\frac{2\pi i n x}{L}} \\
&= \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} a_n(L) e^{\frac{2\pi i n x}{L}} \\
&= \lim_{L\to\infty} f(x) \\
&= f(x),
\end{aligned}$$

as desired. □

5.5.2. Let f and g be the functions defined by

$$f(x) = \chi_{[-1,1]}(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$g(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Although f is not continuous, the integral defining its Fourier transform still makes sense. Show that

$$\hat{f}(\xi) = \frac{\sin(2\pi\xi)}{\pi\xi},$$

$$\hat{g}(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2,$$

with the understanding that $\hat{f}(0) = 2$ and $\hat{g}(0) = 1$.

Solution. Using the definition of the Fourier transform, we have, for all $\xi \neq 0$,

$$\begin{aligned}
\hat{f}(\xi) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx \\
&= \int_{-\infty}^{\infty} \chi_{[-1,1]}(x)e^{-2\pi i x\xi}\,dx \\
&= \int_{-\infty}^{-1} \chi_{[-1,1]}(x)e^{-2\pi i x\xi}\,dx + \int_{-1}^{1} \chi_{[-1,1]}(x)e^{-2\pi i x\xi}\,dx \\
&\quad + \int_{1}^{\infty} \chi_{[-1,1]}(x)e^{-2\pi i x\xi}\,dx \\
&= \int_{-\infty}^{-1} 0e^{-2\pi i x\xi}\,dx + \int_{-1}^{1} 1e^{-2\pi i x\xi}\,dx + \int_{1}^{\infty} 0e^{-2\pi i x\xi}\,dx \\
&= \int_{-1}^{1} e^{-2\pi i x\xi}\,dx \\
&= -\frac{1}{2\pi i\xi}e^{-2\pi i x\xi}\Big|_{-1}^{1} \\
&= -\frac{e^{-2\pi i\xi} - e^{2\pi i\xi}}{2\pi i\xi} \\
&= \frac{1}{\pi\xi}\frac{e^{i(2\pi\xi)} - e^{-i(2\pi\xi)}}{2i} \\
&= \frac{\sin(2\pi\xi)}{\pi\xi}
\end{aligned}$$

and

$$\begin{aligned}
\hat{g}(\xi) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx \\
&= \int_{-\infty}^{-1} f(x)e^{-2\pi i x\xi}\,dx + \int_{-1}^{1} f(x)e^{-2\pi i x\xi}\,dx + \int_{1}^{\infty} f(x)e^{-2\pi i x\xi}\,dx \\
&= \int_{-\infty}^{-1} 0e^{-2\pi i x\xi}\,dx + \int_{-1}^{1} (1 - |x|)e^{-2\pi i x\xi}\,dx + \int_{1}^{\infty} 0e^{-2\pi i x\xi}\,dx \\
&= \int_{-1}^{1} (1 - |x|)e^{-2\pi i x\xi}\,dx \\
&= \int_{-1}^{0} (1 - |x|)e^{-2\pi i x\xi}\,dx + \int_{0}^{1} (1 - |x|)e^{-2\pi i x\xi}\,dx \\
&= \int_{-1}^{0} (1 + x)e^{-2\pi i x\xi}\,dx + \int_{0}^{1} (1 - x)e^{-2\pi i x\xi}\,dx.
\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}
\int_{-1}^{0} (1 + x)e^{-2\pi i x\xi}\,dx &= -\frac{1}{2\pi i\xi}(1 + x)e^{-2\pi i x\xi}\Big|_{-1}^{0} + \frac{1}{2\pi i\xi}\int_{-1}^{0} e^{-2\pi i x\xi}\,dx \\
&= -\frac{1 - 0}{2\pi i\xi} + \frac{1}{4\pi^2\xi^2}e^{-2\pi i x\xi}\Big|_{-1}^{0} \\
&= -\frac{1}{2\pi i\xi} + \frac{1 - e^{2\pi i\xi}}{4\pi^2\xi^2}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 (1-x)e^{-2\pi i x\xi}\,dx &= -\frac{1}{2\pi i\xi}(1-x)e^{-2\pi i x\xi}\Big|_0^1 + \frac{1}{2\pi i\xi}\int_0^1 e^{-2\pi i x\xi}\,dx \\
&= -\frac{0-1}{2\pi i\xi} - \frac{1}{4\pi^2\xi^2}e^{-2\pi i x\xi}\Big|_0^1 \\
&= \frac{1}{2\pi i\xi} - \frac{e^{-2\pi i\xi}-1}{4\pi^2\xi^2} \\
&= \frac{1}{2\pi i\xi} + \frac{1-e^{-2\pi i\xi}}{4\pi^2\xi^2}.
\end{aligned}$$

So we conclude

$$\begin{aligned}
\hat{g}(\xi) &= \int_{-1}^0 (1+x)e^{-2\pi i x\xi}\,dx + \int_0^1 (1-x)e^{-2\pi i x\xi}\,dx \\
&= \left(-\frac{1}{2\pi i\xi} + \frac{1-e^{2\pi i\xi}}{4\pi^2\xi^2}\right) + \left(\frac{1}{2\pi i\xi} + \frac{1-e^{-2\pi i\xi}}{4\pi^2\xi^2}\right) \\
&= \frac{-e^{2\pi i\xi}+2-e^{-2\pi i\xi}}{4\pi^2\xi^2} \\
&= -\frac{e^{2\pi i\xi}-2e^{\pi i\xi}e^{-\pi i\xi}+e^{-2\pi i\xi}}{4\pi^2\xi^2} \\
&= \frac{1}{i^2}\frac{(e^{\pi i\xi}-e^{-\pi i\xi})^2}{(2\pi\xi)^2} \\
&= \frac{1}{(\pi\xi)^2}\left(\frac{e^{\pi i\xi}-e^{-\pi i\xi}}{2i}\right)^2 \\
&= \frac{\sin^2(\pi\xi)}{(\pi\xi)^2} \\
&= \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2,
\end{aligned}$$

as desired. □

5.5.6. The function $e^{-\pi x^2}$ is its own Fourier transform. Generate other functions that (up to a constant multiple) are their own Fourier transforms. What must the constant multiples be? To decide this, prove that $\mathcal{F}^4 = I$. Here, $\mathcal{F}(f) = \hat{f}$ is the Fourier transform, $\mathcal{F}^4 = \mathcal{F}\circ\mathcal{F}\circ\mathcal{F}\circ\mathcal{F}$, and I is the identity operator $(If)(x) = f(x)$.

Solution. □

5.5.21. Suppose that f is continuous on $\mathbb{R}$. Show that f and $\hat{f}$ cannot both be compactly supported unless $f = 0$. This can be viewed in the same spirit as the uncertainty principle.

Solution. Suppose by contradiction that both f and $\hat{f}$ are compactly supported and nonzero. Since f is compactly supported, there exists $K > 0$ such that we have $f(x) = 0$ for all $|x| \geq K$. Likewise, since $\hat{f}$ is compactly supported, there exists $N > 0$ such that we have $\hat{f}(\xi) = 0$ for all $|x| \geq N$. In particular, we have $\hat{f}(n) = 0$ for all integers n satisfying $|n| \geq N$. Now, we can use the Fourier series representation of f to obtain, for any $L > 0$,

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{\frac{2\pi i n x}{L}} \\
&= \sum_{n=-N}^{N} \hat{f}(n)e^{\frac{2\pi i n x}{L}} + \sum_{|n|\geq N} \hat{f}(n)e^{\frac{2\pi i n x}{L}} \\
&= \sum_{n=-N}^{N} \hat{f}(n)e^{\frac{2\pi i n x}{L}} + \sum_{|n|\geq N} 0e^{\frac{2\pi i n x}{L}} \\
&= \sum_{n=-N}^{N} \hat{f}(n)e^{\frac{2\pi i n x}{L}},
\end{aligned}$$

whose final expression tells us that f is a trigonometric polynomial that vanishes on $[\min\{-K,-N\},\max\{-K,-N\}] \cup [\min\{K,N\},\max\{K,N\}]$. But this contradicts the assumption that f vanishes ona finite number of points. So we conclude $f = 0$. □

6.6.7. Consider the time-dependent heat equation in $\mathbb{R}^d$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2},$$

where $t > 0$, with boundary values $u(x, 0) = f(x)$ for $f \in \mathcal{S}(\mathbb{R}^d)$. If

$$\begin{aligned} \mathcal{H}_t^{(d)}(x) &= \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} \\ &= \int_{\mathbb{R}^d} e^{-4\pi^2 t|\xi|^2} e^{2\pi i x\cdot\xi}\, d\xi \end{aligned}$$

is the d-dimensional heat kernel, show that the convolution

$$u(x, t) = (f * \mathcal{H}_t^{(d)})(x)$$

is infinitely differentiable when $x \in \mathbb{R}^d$ and $t > 0$. Moreover, u solves the heat equation, and is continuous up to the boundary $t = 0$ with $u(x, 0) = f(x)$.

Solution. First, we notice that $\mathcal{H}_t^{(d)}(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$ is infinitely differentiable for all $x \in \mathbb{R}^d$ and $t > 0$, which implies in particular that $\mathcal{H}_t^{(d)}$ is uniformly continuous on $\mathbb{R}^d$: for every $\epsilon > 0$, there exists $\delta > 0$ such that, if $x, y \in \mathbb{R}^d$ satisfies $|x - y| < \delta$, then

$$|\mathcal{H}_t^{(d)}(x) - \mathcal{H}_t^{(d)}(y)| < \epsilon.$$

We can show that u is infinitely differentiable by establishing, for all multiindices α with $|\alpha| = k$, where $k \geq 0$ is an integer,

$$\begin{aligned} D^\alpha u(x, t) &= D^\alpha (f * \mathcal{H}_t^{(d)})(x) \\ &= (f * D^\alpha \mathcal{H}_t^{(d)})(x). \end{aligned}$$

We proceed by induction. For the base case, we would like to prove

$$\nabla(f * \mathcal{H}_t^{(d)})(x) = (f * \nabla\mathcal{H}_t^{(d)})(x).$$

Let $h \in B(0, 1) \subset \mathbb{R}^d$; that is, $h \in \mathbb{R}^d$ satisfies $|h| < 1$. Then, for all $y \in \mathbb{R}^d$, we have

$$\begin{aligned} \mathcal{H}_t^{(d)}(x + h - y) - \mathcal{H}_t^{(d)}(x - y) - \nabla\mathcal{H}_t^{(d)}(x - y) \cdot h &= \int_0^1 \frac{\partial}{\partial s}\mathcal{H}_t^{(d)}(x + sh - y)\, ds - \nabla\mathcal{H}_t^{(d)}(x - y) \cdot h \int_0^1 1\, ds \\ &= \int_0^1 \frac{\partial}{\partial s}\mathcal{H}_t^{(d)}(x + sh - y) - \nabla\mathcal{H}_t^{(d)}(x - y) \cdot h\, ds \\ &= \int_0^1 \nabla\mathcal{H}_t^{(d)}(x + sh - y) \cdot h - \nabla\mathcal{H}_t^{(d)}(x - y) \cdot h\, ds \\ &= \int_0^1 (\nabla\mathcal{H}_t^{(d)}(x + sh - y) - \nabla\mathcal{H}_t^{(d)}(x - y)) \cdot h\, ds, \end{aligned}$$

which implies

$$\begin{aligned} |\mathcal{H}_t^{(d)}(x + h - y) - \mathcal{H}_t^{(d)}(x - y) - \nabla\mathcal{H}_t^{(d)}(x - y) \cdot h| &= \left|\int_0^1 (\nabla\mathcal{H}_t^{(d)}(x + sh - y) - \nabla\mathcal{H}_t^{(d)}(x - y)) \cdot h\, ds\right| \\ &\leq \int_0^1 |(\nabla\mathcal{H}_t^{(d)}(x + sh - y) - \nabla\mathcal{H}_t^{(d)}(x - y)) \cdot h|\, ds \\ &= \int_0^1 |\nabla\mathcal{H}_t^{(d)}(x + sh - y) - \nabla\mathcal{H}_t^{(d)}(x - y)||h|\, ds \\ &< \int_0^1 \epsilon|h|\, ds \\ &= \epsilon|h|. \end{aligned}$$

Now let $E \subset \mathbb{R}^d$ be a fixed compact set large enough such that it satisfies $B(0, 1) \cup \text{supp}(f) \subset E$. Then, for all $y \in \mathbb{R}^d \setminus E$, we have

$$\mathcal{H}_t^{(d)}(x + h - y) - \mathcal{H}_t^{(d)}(x - y) - \nabla\mathcal{H}_t^{(d)}(x - y) \cdot h = 0,$$

which implies, for all $y \in \mathbb{R}^d$

$$|\mathcal{H}_t^{(d)}(x + h - y) - \mathcal{H}_t^{(d)}(x - y) - \nabla\mathcal{H}_t^{(d)}(x - y) \cdot h| < \epsilon|h|\chi_E(y).$$

So, for all $x \in \mathbb{R}^d$, we have

$$
\begin{aligned}
&(f * \mathcal{H}_t^{(d)})(x+h) - (f * \mathcal{H}_t^{(d)})(x) - (f * (\nabla \mathcal{H}_t^{(d)}))(x) \cdot h \\
&\quad = \int_{\mathbb{R}^d} f(x+h-y)g(y)\,dy - \int_{\mathbb{R}^d} f(x-y)g(y)\,dy - \left(\int_{\mathbb{R}^d} \nabla f(x-y)g(y)\,dy\right) \cdot h \\
&\quad = \int_{\mathbb{R}^d} f(x+h-y)g(y) - f(x-y)g(y)\,dy - (\nabla f(x-y)\cdot h)g(y)\,dy \\
&\quad = \int_{\mathbb{R}^d} (f(x+h-y) - f(x-y) - \nabla f(x-y)\cdot h)g(y)\,dy,
\end{aligned}
$$

which implies

$$
\begin{aligned}
|(f * \mathcal{H}_t^{(d)})(x+h) - (f * \mathcal{H}_t^{(d)})(x) - (f * (\nabla \mathcal{H}_t^{(d)}))(x) \cdot h| &= \left|\int_{\mathbb{R}^d} (f(x+h-y) - f(x-y) - \nabla f(x-y)\cdot h)g(y)\,dy\right| \\
&\leq \int_{\mathbb{R}^d} |f(x+h-y) - f(x-y) - \nabla f(x-y)\cdot h||g(y)|\,dy \\
&< \int_{\mathbb{R}^d} \epsilon|h||g(y)|\,dy \\
&= \epsilon|h| \int_{\mathbb{R}^d} |g(y)|\,dy \\
&= \epsilon|h| \left(\int_E |g(y)|\,dy + \int_{\mathbb{R}^d\setminus E} |g(y)|\,dy\right) \\
&= \epsilon|h| \left(\int_E |g(y)|\,dy + \int_{\mathbb{R}^d\setminus E} 0\,dy\right) \\
&= \epsilon|h| \int_E |g(y)|\,dy \\
&= K\epsilon|h|,
\end{aligned}
$$

or equivalently

$$
\frac{|(f * \mathcal{H}_t^{(d)})(x+h) - (f * \mathcal{H}_t^{(d)})(x) - (f * (\nabla \mathcal{H}_t^{(d)}))(x) \cdot h|}{|h|} < K\epsilon.
$$

where we set $K := \int_E |g(y)|\,dy < \infty$. Therefore, by definition, the convolution $f * \mathcal{H}_t^{(d)}$ is differentiable with

$$
\nabla(f * \mathcal{H}_t^{(d)})(x) = (f * (\nabla \mathcal{H}_t^{(d)}))(x),
$$

completing the base case of our proof by induction. For the induction step, assume

$$
D^\alpha(f * \mathcal{H}_t^{(d)})(x) = (f * D^\alpha \mathcal{H}_t^{(d)})(x).
$$

Then we can apply our argument of the base case to this assumption with D^α in place of ∇, and we can use $D^{\alpha+1} = \nabla D^\alpha$ to obtain

$$
D^{\alpha+1}(f * \mathcal{H}_t^{(d)})(x) = (f * D^{\alpha+1}\mathcal{H}_t^{(d)})(x).
$$

which completes our proof by induction. □