MATH 147 Discussion
Quiz 1 Solutions
January 22, 2021

Directions: Write your solutions to each question on a separate sheet of paper. Once you are finished with the quiz, take pictures of your solutions to each solution **separately**, and submit your quiz solutions on Crowdmark, separated by question (Q1, Q2, Q3). Please note that you must submit your quiz by 1:10 p.m. deadline, or I will not accept your quiz submission.

(5pts) 1. Tell me a bit about yourself. (This is the only non-math question of this quiz.)

(1pt) (a) What is your full name (first and last)?

(1pt) (b) What is your current major?

(1pt) (c) What year of study are you in? (junior, senior, graduate, etc.)

(1pt) (d) What is your favorite application of Fourier analysis?

(1pt) (e) Draw a really quick portrait sketch of French mathematician and physicist Joseph Fourier. You may use **this image** for reference.

Solution. Answers to parts (a)-(e) may vary. □

(5pts) 2. Let f be the function defined on $[0, 2\pi]$ by

$$f(x) = x\sin(x).$$

Show that the n^{th} Fourier coefficient is

$$\hat{f}(n) = \begin{cases} \frac{1}{n^2-1} & \text{if } n \neq \pm 1, \\ -\frac{1}{4} + \frac{\pi}{2}i, & \text{if } n = -1, \\ -\frac{1}{4} - \frac{\pi}{2}i & \text{if } n = 1 \end{cases}$$

for all integers n.

Suggestion: Use $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$.

Solution. First, we suppose $n \neq \pm 1$. We have

$$\begin{aligned} 4\pi i \hat{f}(n) &= 4\pi i \frac{1}{2\pi} \int_0^{2\pi} x \sin(x) e^{-inx}\, dx \\ &= 2i \int_0^{2\pi} x \frac{e^{ix} - e^{-ix}}{2i} e^{-inx}\, dx \\ &= \int_0^{2\pi} x e^{-i(n-1)x}\, dx - \int_0^{2\pi} x e^{-i(n+1)x}\, dx. \end{aligned}$$

If we assume $n \neq 1$, then we can use integration by parts to obtain

$$\begin{aligned}
\int_0^{2\pi} x e^{-i(n-1)x}\,dx &= -\frac{1}{i(n-1)} x e^{-i(n-1)x}\Big|_0^{2\pi} + \frac{1}{i(n-1)}\int_0^{2\pi} e^{-i(n-1)x}\,dx \\
&= \frac{i}{n-1}(2\pi - 0) - \frac{1}{i^2(n-1)^2} e^{-i(n-1)x}\Big|_0^{2\pi} \\
&= \frac{2\pi i}{n-1} + \frac{1}{(n-1)^2}(1-1) \\
&= \frac{2\pi i}{n-1}.
\end{aligned}$$

If we assume $n \neq -1$, then by similar reasoning we obtain

$$\int_0^{2\pi} x e^{-i(n-1)x}\,dx = \frac{2\pi i}{n+1}.$$

So we obtain

$$\begin{aligned}
4\pi i \hat{f}(n) &= \int_0^{2\pi} x e^{-i(n-1)x}\,dx - \int_0^{2\pi} x e^{-i(n+1)x}\,dx \\
&= \frac{2\pi i}{n-1} - \frac{2\pi i}{n+1} \\
&= \frac{4\pi i}{n^2-1},
\end{aligned}$$

or equivalently

$$\hat{f}(n) = \frac{1}{n^2-1}$$

for all integers $n \neq \pm 1$. Next, we can use integration by parts to also obtain

$$\begin{aligned}
4\pi i \hat{f}(1) &= 4\pi i \frac{1}{2\pi}\int_0^{2\pi} x \sin(x) e^{-ix}\,dx \\
&= 2i \int_0^{2\pi} x \frac{e^{ix} - e^{-ix}}{2i} e^{-ix}\,dx \\
&= \int_0^{2\pi} x\,dx - \int_0^{2\pi} x e^{-2ix}\,dx \\
&= \frac{x^2}{2}\Big|_0^{2\pi} - \left(-\frac{1}{2i} x e^{-2ix}\Big|_0^{2\pi} - \frac{1}{2i}\int_0^{2\pi} e^{-2ix}\,dx \right) \\
&= \frac{4\pi^2 - 0}{2} - \left(\frac{1}{2} i(2\pi - 0) - \frac{1}{4i^2} e^{-2ix}\Big|_0^{2\pi} \right) \\
&= 2\pi^2 - \left(\pi i + \frac{1}{4}(1-1) \right) \\
&= 2\pi^2 - \pi i
\end{aligned}$$

and, by similar reasoning,

$$4\pi i \hat{f}(-1) = -2\pi^2 - \pi i,$$

which imply

$$\hat{f}(1) = -\frac{1}{4} - \frac{\pi}{2}i,$$
$$\hat{f}(-1) = -\frac{1}{4} + \frac{\pi}{2}i,$$

respectively. □

(5pts) 3. Express $f(x) = x\sin(x)$ as a Fourier series in exponential form. Then convert the series into its sine-cosine form.

Solution. The Fourier series of $f(x) = x\sin(x)$ in exponential form is

$$\begin{aligned} f(x) &\sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx} \\ &= \hat{f}(-1)e^{i(-1)x} + \hat{f}(1)e^{i(1)x} + \sum_{\substack{n\neq\pm1 \\ n\in\mathbb{Z}}} \hat{f}(n)e^{inx} \\ &= \boxed{\left(-\frac{1}{4} + \frac{\pi}{2}i\right)e^{-ix} + \left(-\frac{1}{4} - \frac{\pi}{2}i\right)e^{ix} + \sum_{\substack{n\neq\pm1 \\ n\in\mathbb{Z}}} \frac{1}{n^2-1}e^{inx}}. \end{aligned}$$

Next, we will convert our Fourier series into its sine-cosine form. The sum of the first two terms is

$$\begin{aligned} \left(-\frac{1}{4} + \frac{\pi}{2}i\right)e^{-ix} + \left(-\frac{1}{4} - \frac{\pi}{2}i\right)e^{ix} &= -\frac{1}{4}(e^{ix} + e^{-ix}) - \frac{\pi}{2}i(e^{ix} - e^{-ix}) \\ &= -\frac{1}{2}\frac{e^{ix} + e^{-ix}}{2} - \pi i^2 \frac{e^{ix} - e^{-ix}}{2i} \\ &= -\frac{1}{2}\cos(x) + \pi\sin(x), \end{aligned}$$

and the last term is

$$\begin{aligned} \sum_{\substack{n\neq\pm1 \\ n\in\mathbb{Z}}} \frac{1}{n^2-1}e^{inx} &= \sum_{n=-\infty}^{-2} \frac{1}{n^2-1}e^{inx} + \frac{1}{0^2-1}e^{i(0)x} + \sum_{n=2}^{\infty} \frac{1}{n^2-1}e^{inx} \\ &= \sum_{n=2}^{\infty} \frac{1}{(-n)^2-1}e^{i(-n)x} - 1 + \sum_{n=2}^{\infty} \frac{1}{n^2-1}e^{inx} \\ &= -1 + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\frac{e^{inx} + e^{-inx}}{2} \\ &= -1 + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos(nx). \end{aligned}$$

So we obtain

$$
\begin{aligned}
f(x) &\sim \left(-\frac{1}{4} - \frac{\pi}{2}i\right) e^{-ix} + \left(-\frac{1}{4} + \frac{\pi}{2}i\right) e^{ix} + \sum_{\substack{n \neq \pm 1 \\ n \in \mathbb{Z}}} \frac{1}{n^2 - 1} e^{inx} \\
&= \left(-\frac{1}{2}\cos(x) + \pi \sin(x)\right) + \left(-1 + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1}\cos(nx)\right) \\
&= \boxed{-1 - \frac{1}{2}\cos(x) + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1}\cos(nx) + \pi \sin(x)},
\end{aligned}
$$

which is the Fourier series in its sine-cosine form. □