MATH 147 Discussion
Quiz 2 Solutions
February 12, 2021

Directions: Write your solutions to each question on a separate sheet of paper. Once you are finished with the quiz, take pictures of your solutions to each question **separately**, and submit your quiz solutions on Crowdmark, separated by question (Q1, Q2, Q3). Please note that you must submit your quiz by 1:10 p.m. deadline, unless I give a time extension to everyone.

(5pts) 1. Unscramble the following anagrams of the last names of mathematicians relevant to the field of Fourier analysis.

(0.5pts) (a) LIERDITCH **DIRICHLET**

(0.5pts) (b) OURFIRE **FOURIER**

(0.5pts) (c) GREENLED **LEGENDRE**

(0.5pts) (d) CAPELLA **LAPLACE**

(0.5pts) (e) LEPRECHALN **PLANCHEREL**

(0.5pts) (f) SWATCHRZ **SCHWARTZ**

(0.5pts) (g) POISONS **POISSON**

(0.5pts) (h) IMANNER **RIEMANN**

(0.5pts) (i) SAVPEARL **PARSEVAL**

(0.5pts) (j) IDSLAPU **LAPIDUS**

(10pts) 2. Consider the vector space $\mathcal{R}$, the set of complex-valued Riemann integrable functions on $[0, 2\pi]$, equipped with the inner product

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}\, dx$$

and its associated norm

$$\|f\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2\, dx \right)^{\frac{1}{2}}$$

for any $f, g \in \mathcal{R}$. Prove the Cauchy-Schwarz inequality

$$|(f, g)| \le \|f\| \|g\|$$

and the triangle inequality

$$\|f + g\| \le \|f\| + \|g\|.$$

Proof. First, we will prove the Cauchy-Schwarz inequality. To this end, note that, if we assume $a, b \in R$, then the true statement

$$(a - b)^2 \ge 0$$

is algebraically equivalent to

$$ab \le \frac{a^2 + b^2}{2}$$

We also recall the triangle inequality for integrals, which states

$$\left|\int_0^{2\pi} h(x)\,dx\right| \le \int_0^{2\pi} |h(x)|\,dx$$

for any $h \in \mathcal{R}$. We can apply these inequalities to obtain, for all nonzero $f, g \in \mathcal{R}$,

$$\begin{aligned}
\frac{|(f,g)|}{\|f\|\|g\|} &= \frac{1}{\|f\|\|g\|}\left|\frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx\right| \\
&\le \frac{1}{2\pi\|f\|\|g\|}\int_0^{2\pi} |f(x)||\overline{g(x)}|\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(x)|}{\|f\|}\frac{|g(x)|}{\|g\|}\,dx \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\left(\left(\frac{|f(x)|}{\|f\|}\right)^2 + \left(\frac{|g(x)|}{\|g\|}\right)^2\right) dx \\
&= \frac{1}{2}\left(\frac{1}{\|f\|^2}\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx + \frac{1}{\|g\|^2}\frac{1}{2\pi}\int_0^{2\pi} |g(x)|^2\,dx\right) \\
&= \frac{1}{2}\left(\frac{1}{\|f\|^2}\|f\|^2 + \frac{1}{\|g\|^2}\|g\|^2\right) \\
&= \frac{1}{2}(1+1) \\
&= 1 \\
&= \frac{\|f\|\|g\|}{\|f\|\|g\|},
\end{aligned}$$

from which we can multiply both sides by $\|f\|\|g\|$ to conclude

$$|(f,g)| \le \|f\|\|g\|,$$

which is the Cauchy-Schwarz inequality. If we have either $f = 0$ or $g = 0$, then the Cauchy-Schwarz inequality becomes a trivial statement.

Next, we will prove the triangle inequality. Recall that we have

$$z + \bar{z} = 2\,\mathrm{Re}(z)$$

and

$$\mathrm{Re}(z) \le |z|$$

for all $z \in \mathbb{C}$. We can apply these inequalities and the Cauchy-Schwarz inequality to obtain, for all nonzero $f, g \in \mathcal{R}$,

$$\begin{aligned}
\|f+g\|^2 &= \frac{1}{2\pi}\int_0^{2\pi} |f(x)+g(x)|^2\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} (f(x)+g(x))(\overline{f(x)+g(x)})\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} (f(x)+g(x))(\overline{f(x)}+\overline{g(x)})\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{f(x)} + f(x)\overline{g(x)} + g(x)\overline{f(x)} + g(x)\overline{g(x)}\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 + f(x)\overline{g(x)} + g(x)\overline{f(x)} + |g(x)|^2\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx + \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} g(x)\overline{f(x)}\,dx + \frac{1}{2\pi}\int_0^{2\pi} |g(x)|^2\,dx \\
&= \|f\|^2 + (f,g) + (g,f) + \|g\|^2 \\
&= \|f\|^2 + (f,g) + \overline{(f,g)} + \|g\|^2 \\
&= \|f\|^2 + 2\,\mathrm{Re}((f,g)) + \|g\|^2 \\
&\leq \|f\|^2 + 2|(f,g)| + \|g\|^2 \\
&\leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 \\
&= (\|f\| + \|g\|)^2,
\end{aligned}$$

from which we can take the square root of both sides to conclude

$$\|f+g\| \leq \|f\| + \|g\|,$$

which is the triangle inequality. □