MATH 147 Discussion
Quiz 3 Solutions
March 5, 2021

Directions: Write your solution to Question 1 directly on this page[1] and your solutions to Questions 2 and 3 on a separate sheet of paper. Once you are finished with the quiz, take pictures of your solutions to each question **separately**, and submit your quiz solutions on Crowdmark, separated by question (Q1, Q2, Q3). Please note that you must submit your quiz by 1:10 p.m. deadline, unless I give a time extension to everyone.

(5pts) 1. Find in the following word search all ten words of key terms from Chapters 5 and 6 of the Stein and Shakarchi textbook. The words are placed horizontally, vertically, or diagonally, as well as forward or backward. Each word is worth one-half of a point.

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H F D L M A A K M N O I S R E V N I R E I R U O F H Y R L D T J H C S
V W M I R O R W W R J Y O X W X L D Q V V Y K B P B G A N T S Z M P U
B N J A Q O D T R R O C G Z N V H N P Q Z M C E W Q F P N X K N I L B
P U P L T Q M E P V N F D D S G X E B N W E C K M X R I Y K I O G T E
C H N P F H N S R F X X S G L B U V A B A A C O F K U D U E W I V D Q
A F B K W U S E D A Z O Y N V G P A F T P G S B W S J L X R N T R L L
E B D M C W A O G V T S X A A E N O H S K F L F P J D Y H E E U P C H
H P R T K J V A J I H E B O H R B B Z Y M E T D R H M D P K Y L S N D
Z V K A P V U J K N B D D A K U T R J F S G R D Z F Z E X T W O F Z J
H O U P D S L D U M Z O L E N L A R H A U B L N G T I C S R V V M J V
F S M J S I O U Q N S U X V C W T M E B G S P Q E E Z R E R H N E P W
B N D I O W A P I S E A F A H R L A G I P O L V S L U E V Y Z O Z A H
I J A G K F L L K E Y Z K C K A E K V U R M G Q R A A A M M E C U D W
C N Z N I B J B Q Y X D S I W L E A J H O U H N L N D S J J G O G L F
I Y S B B L C M D F A H Q E A N W O S D Y I O K G A X I P K Q W A H Y
Z M L I F N M Y H B Z Y R L V J M G Z E R G E F P T C N P B U P H D Z
N S T X D F T R A N S L A T I O N I N V A R I A N C E G U Y O X F U L
E B B A J F X R A C Q F L H C D V G F G Q L E P J T P J M Y H S V S Q
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Instead of a word bank, the definitions corresponding to the ten words are provided below.

- This process describes going from f to $\hat{f}$.
- This defines the property $\int_{-\infty}^{\infty} f(x-h)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$ for any $h \in \mathbb{R}$.
- What is the name of the set $\mathcal{S}(\mathbb{R})$?
- What is the name of $f \in \mathcal{S}(\mathbb{R})$ defined by $f(x) = e^{-x^2}$?
- This kind of f defined on $\mathbb{R}$ satisfies $|f(x)| \leq \frac{A}{1+x^2}$ for some $A > 0$.
- This kind of f satisfies $\sup_{x\in\mathbb{R}} |x|^k |f^{(\ell)}| < \infty$ for all integers $k, l \geq 0$.
- If $f, g \in \mathcal{S}(\mathbb{R})$, then $(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$.
- This process describes going from $\hat{f}$ to f.
- The solution of the heat equation is $u(x,t) = (f * \mathcal{H}_t)(x)$, where $\mathcal{H}_t$ is called this.
- A function $f \in \mathcal{S}(\mathbb{R}^d)$ for any integer $d \geq 1$ is called this if it depends only on $|x|$.

[1] If for any reason you must write your solution to this question on a separate sheet of paper, please write the (i, j)-entries corresponding to the start and end of a word. For instance, the word `MATH` starts at row 2, column 3 and ends at row 5, column 6; you would write your answer to that word on your separate sheet as `(2,3)->(5,6)`.

(8pts) 2. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \chi_{[0,1]}(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(4pts) (a) Compute the expression of the Fourier transform $\hat{f}$ over $\mathbb{R}$.
Note: Your final answer should be expressed in terms of a sine function.

Solution. Using the definition of the Fourier transform, we have, for all $\xi \neq 0$,

$$\begin{aligned}
\hat{f}(\xi) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx \\
&= \int_{-\infty}^{\infty} \chi_{[0,1]}(x)e^{-2\pi i x\xi}\,dx \\
&= \int_{-\infty}^{0} \chi_{[0,1]}(x)e^{-2\pi i x\xi}\,dx + \int_{0}^{1} \chi_{[0,1]}(x)e^{-2\pi i x\xi}\,dx \\
&\quad + \int_{1}^{\infty} \chi_{[0,1]}(x)e^{-2\pi i x\xi}\,dx \\
&= \int_{-\infty}^{0} 0e^{-2\pi i x\xi}\,dx + \int_{0}^{1} 1e^{-2\pi i x\xi}\,dx + \int_{1}^{\infty} 0e^{-2\pi i x\xi}\,dx \\
&= \int_{0}^{1} e^{-2\pi i x\xi}\,dx \\
&= -\frac{1}{2\pi i\xi}e^{-2\pi i x\xi}\Big|_0^1 \\
&= -\frac{e^{-2\pi i\xi} - 1}{2\pi i\xi} \\
&= \frac{1 - e^{-2\pi i\xi}}{2\pi i\xi}\frac{e^{\pi i\xi}}{e^{\pi i\xi}} \\
&= \frac{e^{i\pi\xi} - e^{-i\pi\xi}}{2i}\frac{1}{\pi\xi e^{\pi i\xi}} \\
&= \sin(\pi\xi)\frac{1}{\pi\xi e^{\pi i\xi}} \\
&= \boxed{\frac{\sin(\pi\xi)}{\pi\xi}e^{-\pi i\xi}},
\end{aligned}$$

and

$$\begin{aligned}\hat{f}(0) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi i x(0)}\,dx \\ &= \int_{-\infty}^{\infty} \chi_{[0,1]}(x)\,dx \\ &= \int_{-\infty}^{0} \chi_{[0,1]}(x)\,dx + \int_{0}^{1} \chi_{[0,1]}(x)\,dx + \int_{1}^{\infty} \chi_{[0,1]}(x)\,dx \\ &= \int_{-\infty}^{0} 0e\,dx + \int_{0}^{1} 1\,dx + \int_{1}^{\infty} 0\,dx \\ &= \int_{0}^{1} 1\,dx \\ &= x|_0^1 \\ &= 1 - 0 \\ &= \boxed{1},\end{aligned}$$

as desired. □

(4pts) (b) Show that $\hat{f}$ is continuous and bounded on $\mathbb{R}$.
Note: You may use without proof any known rules of continuity and the basic properties of sine, cosine, exponential functions, and complex numbers.

Solution. First, we will show that $\hat{f}$ is continuous. Notice that $\sin(x)$ is continuous for all $x \in \mathbb{R}$ and $\frac{1}{x}$ is continuous for all $x \neq 0$. Since a composition of continuous functions is continuous, it follows that $\frac{\sin(x)}{x}$ is continuous for all $x \neq 0$. In other words, $\hat{f}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}$ is continuous for all $\xi \neq 0$. Furthermore, we have $\lim_{\xi \to 0} \hat{f}(\xi) = \hat{f}(0)$, and so we conclude that $\hat{f}$ is continuous on $\mathbb{R}$.

Next, we will show that $\hat{f}$ is bounded on $\mathbb{R}$. Indeed, for all $\xi \neq 0$, we have

$$\begin{aligned}|\hat{f}(\xi)| &= |\hat{f}(\xi)| \\ &= \left|\frac{\sin(\pi\xi)}{\pi\xi} e^{-\pi i \xi}\right| \\ &= \frac{|\sin(\pi\xi)|}{\pi\xi}|e^{-\pi i\xi}| \\ &= \frac{|\sin(\pi\xi)|}{\pi\xi} \cdot 1 \\ &= \frac{|\sin(\pi\xi)|}{\pi\xi} \\ &\leq \frac{\pi\xi}{\pi\xi} \\ &= 1\end{aligned}$$

So we conclude $|\hat{f}(\xi)| \leq 1$ for all $x \in \mathbb{R}$, which means $\hat{f}$ is bounded on $\mathbb{R}$. □

(2pts) 3. We had such an exciting time in Fourier analysis this quarter. What was *your* favorite part of the course?[2]

[2] I like that too.

Solution. Answers may vary. For example, *my* favorite part of this course was helping you guys with the difficult homework problems. :) ∎