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Boole's inequality

Note: Boole's inequality is Theorem 1.3.7 (pages 17-18) of the textbook *Introduction to Mathematical Statistics* (seventh edition) by Robert V. Hogg, Joseph W. McKean, Allen T. Craig. I am following the proof of Theorem 1.3.7 but filling in intermediate steps here, so that the proof is hopefully easier to read. Yes, there is a proof by induction of Boole's inequality, which is shorter and simpler to write than the one presented in the textbook. Please write that induction proof for yourself as an exercise.

Theorem (Boole's Inequality; Theorem 1.3.8 of Hogg, McKean, Craig)**.** *Let $\{C_n\}$ be an arbitrary sequence of events. Then*

$$P\left(\bigcup_{n=1}^{\infty} C_n\right) \leq \sum_{n=1}^{\infty} P(C_n).$$

Proof. Let

$$D_n = \bigcup_{k=1}^{n} C_k,$$

and note that we can write equivalently $D_n = C_1 \cup \cdots \cup C_n$. Then $\{D_n\}$ is an increasing sequence of events; indeed, we have

$$D_1 \subset D_2 \subset D_2 \subset D_4 \subset \cdots,$$

which, by our definition of D_n, is equivalent to saying

$$C_1 \subset C_1 \cup C_2 \subset C_1 \cup C_2 \cup C_2 \subset C_1 \cup C_2 \cup C_3 \cup C_4 \subset \cdots.$$

Also, for all $k = 2, 3, \ldots, n$, we have

$$\begin{aligned} D_k &= C_1 \cup \cdots \cup C_k \\ &= (C_1 \cup \cdots \cup C_{k-1}) \cup C_k \\ &= D_{k-1} \cup C_k. \end{aligned}$$

By the inclusion-exclusion principle (Theorem 1.3.5) and the first axiom of probability, we have

$$\begin{aligned} P(D_k) &= P(D_{k-1} \cup C_k) \\ &= P(D_{k-1}) + P(C_k) - P(D_{k-1} \cap C_k) \\ &\leq P(D_{k-1}) + P(C_k) - 0 \\ &= P(D_{k-1}) + P(C_k), \end{aligned}$$

from which we can subtract $P(D_{k-1})$ from both sides to conclude

$$P(D_k) - P(D_{k-1}) \leq P(C_k) \qquad (*)$$

for all $k = 2, 3, \ldots, n$. Next, observe that, since again $\{D_n\}$ is an increasing sequence of subsets, any union of sets D_n will be the largest set of the increasing sequence; which means we can express

$$\begin{aligned} D_1 \cup \cdots \cup D_n &= C_1 \cup (C_1 \cup C_2) \cup \cdots \cup (C_1 \cup \cdots \cup C_n) \\ &= C_1 \cup \cdots \cup C_n. \end{aligned}$$

Additionally, we can express any union of sets D_n as a *disjoint union* of suitable sets; in other words, we can write

$$D_1 \cup \cdots \cup D_n = D_1 \cup (D_2 \setminus D_1) \cup (D_3 \setminus D_2) \cup \cdots \cup (D_{n-2} \setminus D_{n-1}) \cup (D_{n-1} \setminus D_n).$$

Indeed, the right-hand side of the above equation is a disjoint union because $D_{k-1} \cap (D_k \setminus D_{k-1}) = \varnothing$ for all $k = 2, 3, \ldots, n$. The disjoint union would allow us to apply the third axiom of probability to write

$$P(D_1 \cup (D_2 \setminus D_1) \cup (D_3 \setminus D_2) \cup \cdots \cup (D_{n-2} \setminus D_{n-1}) \cup (D_{n-1} \setminus D_n)) = P(D_1) + P(D_2 \setminus D_1) + \cdots + P(D_{n-1} \setminus D_n).$$

Now, combining our results, we conclude

$$\begin{aligned} C_1 \cup \cdots \cup C_n &= D_1 \cup D_2 \cup \cdots \cup D_n \\ &= D_1 \cup (D_2 \setminus D_1) \cup (D_3 \setminus D_2) \cup \cdots \cup (D_{n-2} \setminus D_{n-1}) \cup (D_{n-1} \setminus D_n), \end{aligned}$$

and so we have

$$
\begin{aligned}
P\left(\bigcup_{k=1}^{n} C_k\right) &= P(C_1 \cup \cdots \cup C_n) \\
&= P(D_1 \cup (D_2 \setminus D_1) \cup (D_3 \setminus D_2) \cup \cdots \cup (D_{n-2} \setminus D_{n-1}) \cup (D_{n-1} \setminus D_n)) \\
&= P(D_1) + P(D_2 \setminus D_1) + \cdots + P(D_{n-1} \setminus D_n) \\
&= P(D_1) + \sum_{k=2}^{n} P(D_k \setminus D_{k-1}) \\
&= P(D_1) + \sum_{k=2}^{n} (P(D_k) - P(D_{k-1})).
\end{aligned}
$$

By $(*)$ from above, we can further obtain

$$
\begin{aligned}
P\left(\bigcup_{k=1}^{n} C_k\right) &= P(D_1) + \sum_{k=2}^{n} (P(D_k) - P(D_{k-1})) \\
&\leq P(D_1) + \sum_{k=2}^{n} P(C_k) \\
&= P(C_1) + \sum_{k=2}^{n} P(C_k) \\
&= \sum_{k=1}^{n} P(C_k).
\end{aligned}
$$

Finally, by Theorem 1.3.6, which allows one to pass the limit inside or outside the probability function P, we conclude

$$
\begin{aligned}
P\left(\bigcup_{k=1}^{\infty} C_k\right) &= P\left(\lim_{n\to\infty} \bigcup_{k=1}^{n} C_k\right) \\
&= \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} C_k\right) \\
&\leq \lim_{n\to\infty} \sum_{k=1}^{n} P(C_k) \\
&= \sum_{k=1}^{\infty} P(C_k),
\end{aligned}
$$

as desired. $\square$