MATH 149A discussion Ryan Ta
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Boole's inequality

This is another proof of Boole's inequality, one that is done using a proof technique called proof by induction. For your quiz on October 22, you may use the proof by induction, the textbook proof, or any other proof that is valid. Any valid proof that is written 100% correctly will merit full credit for your first quiz score.

Theorem (Boole's Inequality; Theorem 1.3.8 of Hogg, McKean, Craig)**.** *Let $\{C_n\}$ be an arbitrary sequence of events. Then*

$$P\left(\bigcup_{n=1}^{\infty} C_n\right) \le \sum_{n=1}^{\infty} P(C_n).$$

Proof. First, we will establish that the assertion holds true for finite unions and summations; in other words, we will prove that the statement

$$P\left(\bigcup_{k=1}^{n} C_k\right) \le \sum_{k=1}^{n} P(C_k).$$

for any positive integer n holds true. For this, we will proceed by induction. For the base step, we will prove that the statement for $n = 1$ holds true. A union of one set C_1 only is the set C_1 itself. Furthermore, any summation of one number $P(C_1)$ is the number $P(C_1)$ itself. So the assertion for one set C_1 becomes

$$P(C_1) \le P(C_1),$$

which is clearly true. This completes the proof of the base step. For the induction step, we assume that the statement for $n = m$ holds true; in other words, we assume that the statement

$$P\left(\bigcup_{k=1}^{m} C_k\right) \le \sum_{k=1}^{m} P(C_k)$$

holds true. With this assumption, we will prove that the statement for $n = m + 1$ holds true; in other words, we will prove the statement

$$P\left(\bigcup_{k=1}^{m+1} C_k\right) \le \sum_{k=1}^{m+1} P(C_k).$$

Indeed, using the inclusion-exclusion principle, the first axiom of probability, and what we assumed in this induction step, we have

$$\begin{aligned}
P\left(\bigcup_{k=1}^{m+1} C_k\right) &= P\left(\bigcup_{k=1}^{m} C_k \cup C_{m+1}\right) \\
&= P\left(\bigcup_{k=1}^{m} C_k\right) + P(C_{m+1}) - P\left(\bigcup_{k=1}^{m} C_k \cap C_{m+1}\right) \\
&\le P\left(\bigcup_{k=1}^{m} C_k\right) + P(C_{m+1}) - 0 \\
&= P\left(\bigcup_{k=1}^{m} C_k\right) + P(C_{m+1}) \\
&\le \sum_{k=1}^{m} P(C_k) + P(C_{m+1}) \\
&= \sum_{k=1}^{m+1} P(C_k).
\end{aligned}$$

This completes our proof by induction, and establishes that the statement

$$P\left(\bigcup_{k=1}^{n} C_k\right) \le \sum_{k=1}^{n} P(C_k).$$

holds for all positive integers n. Finally, by Theorem 1.3.6, which allows one to pass the limit inside or outside the probability function P, we conclude

$$\begin{aligned}
P\left(\bigcup_{k=1}^{\infty} C_k\right) &= P\left(\lim_{n\to\infty} \bigcup_{k=1}^{n} C_k\right) \\
&= \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} C_k\right) \\
&\leq \lim_{n\to\infty} \sum_{k=1}^{n} P(C_k) \\
&= \sum_{k=1}^{\infty} P(C_k),
\end{aligned}$$

as desired. $\square$