MATH 149A discussion | Ryan Ta
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Chebyshev's inequality

Note: Chebyshev's inequality is Theorem 1.10.3 (page 70) of the textbook *Introduction to Mathematical Statistics* (seventh edition) by Robert V. Hogg, Joseph W. McKean, Allen T. Craig. I am following the proof of Theorem 1.10.3 but filling in intermediate steps here, so that the proof is hopefully easier to read.

Theorem (Chebyshev's inequality; Theorem 1.10.3 of Hogg, McKean, Craig)**.** *Let the random variable X have a distribution of probability about which we assume only that there is a finite variance σ^2; this implies that the mean $\mu = E(X)$ exists. Then, for every $k > 0$, we have*

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

Proof. Markov's inequality states: If $u(X)$ be a nonnegative function of the random variable X such that $E[u(X)]$ exists, then, for every positive constant c, we have

$$P[u(X) \geq c] \leq \frac{E[u(X)]}{c}.$$

If we let $u(X) = (X-\mu)^2$ and $c = k^2\sigma^2$, then Markov's inequality implies

$$\begin{aligned} P[(X-\mu)^2 \geq k^2\sigma^2] &= P[u(X) \geq c] \\ &\leq \frac{E[u(X)]}{c} \\ &= \frac{E[(X-\mu)^2]}{k^2\sigma^2}. \end{aligned}$$

As we recall

$$\begin{aligned} E[(X-\mu)^2] &= E(X^2 - 2\mu X + \mu^2) \\ &= E(X^2) - 2\mu E(X) + E(\mu^2) \\ &= E(X^2) - 2\mu\mu + \mu^2 \\ &= E(X^2) - \mu^2 \\ &= E(X^2) - (E(X))^2 \\ &= \mathrm{Var}(X) \\ &= \sigma^2. \end{aligned}$$

Finally, we note that we have $|X-\mu| \geq k\sigma$ if and only if we have $(X-\mu)^2 \geq k^2\sigma^2$, which implies the set equality

$$(|X-\mu| \geq k\sigma) = ((X-\mu)^2 \geq k^2\sigma^2).$$

Therefore, we obtain

$$\begin{aligned} P[|X-\mu| \geq k\sigma] &= P[(X-\mu)^2 \geq k^2\sigma^2] \\ &\leq \frac{E[(X-\mu)^2]}{k^2\sigma^2} \\ &= \frac{\sigma^2}{k^2\sigma^2} \\ &= \frac{1}{k^2}, \end{aligned}$$

which is Chebyshev's inequality, as desired. □