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Markov's inequality

Note: Markov's inequality is Theorem 1.10.2 (pages 68-69) of the textbook *Introduction to Mathematical Statistics* (seventh edition) by Robert V. Hogg, Joseph W. McKean, Allen T. Craig. I am following the proof of Theorem 1.10.2 but filling in intermediate steps here, so that the proof is hopefully easier to read.

Theorem (Markov's inequality; Theorem 1.10.2 of Hogg, McKean, Craig)**.** *Let $u(X)$ be a nonnegative function of the random variable X. If $E[u(X)]$ exists, then, for every positive constant c, we have*

$$P[u(X) \geq c] \leq \frac{E[u(X)]}{c}.$$

Proof. First, we will assume that X is a continuous random variable. Let $f_X(x)$ denote the pdf of X. Since the expectation of $u(X)$ exists, we have

$$\begin{aligned} |E[u(X)]| &= \left| \int_{-\infty}^{\infty} u(x) f_X(x)\, dx \right| \\ &\leq \int_{-\infty}^{\infty} |u(x)| f_X(x)\, dx \\ &< \infty. \end{aligned}$$

As $f_X(x)$ denotes the pdf of X, by definition $f_X(x)$ is nonnegative. As $u(X)$ is also nonnegative, we have

$$\int_{\{x \in \mathbb{R} : u(x) < c\}} u(x) f_X(x)\, dx \geq 0.$$

This means we get

$$\begin{aligned} E[u(X)] &= \int_{-\infty}^{\infty} u(x) f_X(x)\, dx \\ &= \int_{\{x \in \mathbb{R} : u(x) \geq c\}} u(x) f_X(x)\, dx + \int_{\{x \in \mathbb{R} : u(x) < c\}} u(x) f_X(x)\, dx \\ &\geq \int_{\{x \in \mathbb{R} : u(x) \geq c\}} u(x) f_X(x)\, dx + 0 \\ &= \int_{\{x \in \mathbb{R} : u(x) \geq c\}} u(x) f_X(x)\, dx \\ &\geq \int_{\{x \in \mathbb{R} : u(x) \geq c\}} c f_X(x)\, dx \\ &= c \int_{\{x \in \mathbb{R} : u(x) \geq c\}} f_X(x)\, dx \\ &= c P[u(X) \geq c], \end{aligned}$$

which is equivalent to $P[u(X) \geq c] \leq \frac{E[u(X)]}{c}$.

Next, we will assume that X is a discrete random variable. Let $p_X(x)$ denote the pmf of X. Since the expectation of $u(X)$ exists, we have

$$\begin{aligned} |E[u(X)]| &= \left| \sum_x u(x) p_X(x) \right| \\ &\leq \sum_x |u(x)| p_X(x) \\ &< \infty. \end{aligned}$$

As $p_X(x)$ denotes the pmf of X, by definition $p_X(x)$ is nonnegative. As $u(X)$ is also nonnegative, we have

$$\sum_{\substack{x \\ u(x) < c}} u(x) p_X(x) \geq 0.$$

This means we get

$$
\begin{aligned}
E[u(X)] &= \sum_x u(x)p_X(x) \\
&= \sum_{\substack{x \\ u(x)\geq c}} u(x)p_X(x) + \sum_{\substack{x \\ u(x)<c}} u(x)p_X(x) \\
&\geq \sum_{\substack{x \\ u(x)\geq c}} u(x)p_X(x) + 0 \\
&= \sum_{\substack{x \\ u(x)\geq c}} u(x)p_X(x) \\
&\geq \sum_{\substack{x \\ u(x)\geq c}} cp_X(x) \\
&= c \sum_{\substack{x \\ u(x)\geq c}} p_X(x) \\
&= cP[u(X) \geq c],
\end{aligned}
$$

which is equivalent to $P[u(X) \geq c] \leq \frac{E[u(X)]}{c}$. □