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Examples addressed in discussion that may be similar to a question on Quiz 2

Our first example will make use of double integrals in Cartesian coordinates.

Example. *Let* (X, Y) *be two variables whose joint pdf has the form*

$$f_{X,Y}(x,y) = \begin{cases} c(x^2+y^2) & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

where c *is some positive constant. First, we will find the value of* c*. The integral of the pdf, over the whole region, should be* 1*. We have*

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy \\ &= \int_0^1\int_0^1 c(x^2+y^2)\,dx\,dy \\ &= c\int_0^1\int_0^1 x^2+y^2\,dx\,dy \\ &= c\int_0^1 \left(\frac{x^3}{3}+xy^2\right)\Big|_0^1\,dy \\ &= c\int_0^1 \left(\left(\frac{(1)^3}{3}+(1)y^2\right)-\left(\frac{(0)^3}{3}+(0)y^2\right)\right)d\theta \\ &= c\int_0^1 \frac{1}{3}+y^2\,d\theta \\ &= c\left(\frac{y}{3}+\frac{y^3}{3}\right)\Big|_0^1 \\ &= c\left(\left(\frac{(1)}{3}+\frac{(1)^3}{3}\right)-\left(\frac{(0)}{3}+\frac{(0)^3}{3}\right)\right) \\ &= \frac{2}{3}c. \end{aligned}$$

This means $c = \frac{3}{2}$*. Next, we will find the expectations of* X*,* Y*, and* $X+Y$*. We have*

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dx\,dy \\ &= \int_0^1\int_0^1 x(c(x^2+y^2))\,dx\,dy \\ &= c\int_0^1\int_0^1 x^3+xy^2\,dx\,dy \\ &= c\int_0^1 \left(\frac{x^4}{4}+\frac{x^2y^2}{2}\right)\Big|_0^1\,dy \\ &= c\int_0^1 \left(\left(\frac{(1)^4}{4}+\frac{(1)^2y^2}{2}\right)-\left(\frac{(0)^4}{4}+\frac{(0)^2y^2}{2}\right)\right)dy \\ &= c\int_0^1 \frac{1}{4}+\frac{y^2}{2}\,dy \\ &= c\left(\frac{y}{4}+\frac{y^3}{6}\right)\Big|_0^1 \\ &= c\left(\left(\frac{(1)}{4}+\frac{(1)^3}{6}\right)-\left(\frac{(0)}{4}+\frac{(0)^3}{6}\right)\right) \\ &= \frac{5}{12}c \\ &= \frac{5}{12}\frac{3}{2} \\ &= \frac{5}{8} \end{aligned}$$

and

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,dx\,dy \\
&= \int_0^1\int_0^1 y(c(x^2+y^2))\,dx\,dy \\
&= c\int_0^1\int_0^1 x^2y+y^3\,dx\,dy \\
&= c\int_0^1 \left(\frac{x^3y}{3}+xy^3\right)\Big|_0^1\,dy \\
&= c\int_0^1 \left(\left(\frac{(1)^3y}{3}+(1)y^3\right)-\left(\frac{(0)^3y}{3}+(0)y^3\right)\right)dy \\
&= c\int_0^1 \frac{y}{3}+y^3\,dy \\
&= c\left(\frac{y^2}{6}+\frac{y^4}{4}\right)\Big|_0^1 \\
&= c\left(\left(\frac{(1)^2}{6}+\frac{(1)^4}{4}\right)-\left(\frac{(0)^2}{6}+\frac{(0)^4}{4}\right)\right) \\
&= \frac{5}{12}c \\
&= \frac{5}{12}\frac{3}{2} \\
&= \frac{5}{8}.
\end{aligned}$$

By linearity of expectation, we have

$$\begin{aligned}
E(X+Y) &= E(X)+E(Y) \\
&= \frac{5}{8}+\frac{5}{8} \\
&= \frac{5}{4}
\end{aligned}$$

as desired.

Our second example will make use of double integrals in polar coordinates.

Example. *Let (X,Y) be two variables whose joint pdf has the form*

$$f_{X,Y}(x,y) = \begin{cases} c(x^2+y^2) & \text{if } x^2+y^2 \le \sqrt{\pi}, y \ge 0 \\ 0 & \text{otherwise,} \end{cases}$$

where c is some positive constant. First, we will find the value of c. The integral of the pdf, over the whole region, should be 1. *Writing $x = r\cos\theta$ and $y = r\sin\theta$, we have*

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy \\
&= \int_0^{\pi}\int_0^{\sqrt[4]{\pi}} (cr^2)r\,dr\,d\theta \\
&= c\int_0^{\pi}\int_0^{\sqrt[4]{\pi}} r^3\,dr\,d\theta \\
&= c\int_0^{\pi} \frac{r^4}{4}\Big|_0^{\sqrt[4]{\pi}}\,d\theta \\
&= c\int_0^{\pi}\left(\frac{(\sqrt[4]{\pi})^4}{4}-\frac{(0)^4}{4}\right)d\theta \\
&= c\int_0^{\pi}\frac{\pi}{4}\,d\theta \\
&= \frac{\pi}{4}c\theta\Big|_0^{\pi} \\
&= \frac{\pi}{4}c(\pi-0) \\
&= \frac{\pi^2}{4}c.
\end{aligned}$$

This means $c = \frac{4}{\pi^2}$*. Next, we will find the expectations of* X*,* Y*, and* $X+Y$*. Writing* $x = r\cos\theta$ *and* $y = r\sin\theta$*, we have*

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dx\,dy \\
&= \int_0^{\pi}\int_0^{\sqrt[4]{\pi}} (r\cos\theta)(cr^2)r\,dr\,d\theta \\
&= c\int_0^{\pi}\int_0^{\sqrt[4]{\pi}} r^4\cos\theta\,dr\,d\theta \\
&= c\int_0^{\pi} \left.\frac{r^5}{5}\right|_0^{\sqrt[4]{\pi}} \cos\theta\,d\theta \\
&= c\int_0^{\pi} \left(\frac{(\sqrt[4]{\pi})^5}{5} - \frac{(0)^5}{5}\right)\cos\theta\,d\theta \\
&= \frac{\pi^{\frac{5}{4}}}{5}c\int_0^{\pi}\cos\theta\,d\theta \\
&= \left.\frac{\pi^{\frac{5}{4}}}{5}c\sin\theta\right|_0^{\pi} \\
&= \frac{\pi^{\frac{5}{4}}}{5}\frac{4}{\pi^2}(\sin\pi - \sin 0) \\
&= \frac{4}{\pi^{\frac{3}{4}}}(0-0) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,dx\,dy \\
&= \int_0^{\pi}\int_0^{\sqrt[4]{\pi}} (r\sin\theta)(cr^2)r\,dr\,d\theta \\
&= c\int_0^{\pi}\int_0^{\sqrt[4]{\pi}} r^4\sin\theta\,dr\,d\theta \\
&= c\int_0^{\pi} \left.\frac{r^5}{5}\right|_0^{\sqrt[4]{\pi}} \sin\theta\,d\theta \\
&= c\int_0^{\pi} \left(\frac{(\sqrt[4]{\pi})^5}{5} - \frac{(0)^5}{5}\right)\sin\theta\,d\theta \\
&= \frac{\pi^{\frac{5}{4}}}{5}c\int_0^{\pi}\sin\theta\,d\theta \\
&= \left.\frac{\pi^{\frac{5}{4}}}{5}c(-\cos\theta)\right|_0^{\pi} \\
&= \frac{\pi^{\frac{5}{4}}}{5}c((-\cos\pi) - (-\cos 0)) \\
&= \frac{\pi^{\frac{5}{4}}}{5}\frac{4}{\pi^2}((-(-1)) - (-1)) \\
&= \frac{8}{5\pi^{\frac{3}{4}}}.
\end{aligned}$$

By linearity of expectation, we have

$$\begin{aligned}
E(X+Y) &= E(X) + E(Y) \\
&= 0 + \frac{8}{5\pi^{\frac{3}{4}}} \\
&= \frac{8}{5\pi^{\frac{3}{4}}},
\end{aligned}$$

as desired.