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Quiz 2 solutions
Section 11
This quiz is worth 13 points.

(4pts) 1. State and prove either Markov's Inequality or Chebyshev's Inequality.

Note: If you decide to prove Markov's inequality, you may choose to prove either the continuous case or the discrete case. If you decide to prove Chebyshev's Inequality, you may cite Markov's Inequality without its proof.

Statement of Markov's inequality (Theorem 1.10.2 of Hogg, McKean, Craig)**.** Let $u(X)$ be a nonnegative function of the random variable X. If $E[u(X)]$ exists, then for every positive constant c, we have

$$P[u(X) \geq c] \leq \frac{E[u(X)]}{c}.$$

Proof. See the discussion notes for November 19 or Theorem 1.10.2 of the textbook. □

Statement of Chebyshev's inequality (Theorem 1.10.3 of Hogg, McKean, Craig)**.** Let the random variable X have a distribution of probability about which we assume only that there is a finite variance σ^2. Then, for every $k > 0$, we have

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

Proof. See the discussion notes for November 19 or Theorem 1.10.3 of the textbook. □

(9pts) 2. Let (X, Y) be two random variables whose joint pdf has the form

$$f_{X,Y}(x, y) = \begin{cases} c^2 - (x^2 + y^2)^2 & \text{if } x^2 + y^2 \leq c, x \geq 0, y \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where c is some positive constant.

(4pts) (a) Determine the value of c.

Solution. The integral of the pdf, over the whole region, should be 1. Writing $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dx\,dy \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{c}} (c^2 - (r^2)^2)r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{c}} c^2 r - r^5\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{c^2}{2}r^2 - \frac{1}{6}r^6\right)\bigg|_0^{\sqrt{c}}\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(\left(\frac{c^2}{2}(\sqrt{c})^2 - \frac{1}{6}(\sqrt{c})^6\right) - \left(\frac{c^2}{2}(0)^6 - \frac{1}{6}(0)^6\right)\right)d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{3}c^3\,d\theta \\
&= \frac{1}{3}c^3\theta\bigg|_0^{\frac{\pi}{2}} \\
&= \frac{1}{3}c^3\left(\frac{\pi}{2} - 0\right) \\
&= \frac{\pi}{6}c^3.
\end{aligned}$$

This means $c = (\frac{6}{\pi})^{\frac{1}{3}}$. □

(5pts) (b) With the value of c found in part (a), determine the expectations of X, Y, and $X + Y$.

Note: If you were unable to find the value of c in part (a), assume $c = 1$ for your calculations of part (b), even though $c = 1$ is not the correct value.

Solution. Writing $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dx\,dy \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{c}} (r\cos\theta)(c^2-(r^2)^2)r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos\theta \int_0^{\sqrt{c}} c^2r^2 - r^6\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos\theta \left(\frac{c^2}{3}r^3 - \frac{1}{7}r^7\right)\Big|_0^{\sqrt{c}}\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos\theta \left(\left(\frac{c^2}{3}(\sqrt{c})^3 - \frac{1}{7}(\sqrt{c})^7\right) - \left(\frac{c^2}{2}(0)^3 - \frac{1}{7}(0)^7\right)\right)\,d\theta \\
&= \frac{4}{21}c^{\frac{7}{2}}\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta \\
&= \frac{4}{21}c^{\frac{7}{2}}\sin\theta\Big|_0^{\frac{\pi}{2}} \\
&= \frac{4}{21}c^{\frac{7}{2}}\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) \\
&= \frac{4}{21}\left(\left(\frac{6}{\pi}\right)^{\frac{1}{3}}\right)^{\frac{7}{2}}(1-0) \\
&= \boxed{\frac{4}{21}\left(\frac{6}{\pi}\right)^{\frac{7}{6}}}
\end{aligned}$$

and

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,dx\,dy \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{c}} (r\sin\theta)(c^2-(r^2)^2)r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin\theta \int_0^{\sqrt{c}} c^2r^2 - r^6\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin\theta \left(\frac{c^2}{3}r^3 - \frac{1}{7}r^7\right)\Big|_0^{\sqrt{c}}\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin\theta \left(\left(\frac{c^2}{3}(\sqrt{c})^3 - \frac{1}{7}(\sqrt{c})^7\right) - \left(\frac{c^2}{3}(0)^3 - \frac{1}{7}(0)^7\right)\right)\,d\theta \\
&= \frac{4}{21}c^{\frac{7}{2}}\int_0^{\frac{\pi}{2}} \sin\theta\,d\theta \\
&= \frac{4}{21}c^{\frac{7}{2}}(-\cos\theta)\Big|_0^{\frac{\pi}{2}} \\
&= \frac{4}{21}c^{\frac{7}{2}}\left(\left(-\cos\left(\frac{\pi}{2}\right)\right) - (-\cos(0))\right) \\
&= \frac{4}{21}\left(\left(\frac{6}{\pi}\right)^{\frac{1}{3}}\right)^{\frac{7}{2}}((-0)-(-1)) \\
&= \boxed{\frac{4}{21}\left(\frac{6}{\pi}\right)^{\frac{7}{6}}}.
\end{aligned}$$

By linearity of expectation, we have

$$\begin{aligned}
E(X+Y) &= E(X) + E(Y) \\
&= \frac{4}{21}\left(\frac{6}{\pi}\right)^{\frac{7}{6}} + \frac{4}{21}\left(\frac{6}{\pi}\right)^{\frac{7}{6}} \\
&= \frac{8}{21}\left(\frac{6}{\pi}\right)^{\frac{7}{6}},
\end{aligned}$$

as desired. □