MATH 149A discussion
University of California, Riverside

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Chapter 1 exercises of "Introduction to Mathematical Statistics" (seventh edition) by Hogg, McKean, and Craig.

Suggested problems: 1.3.4, 1.3.5, 1.3.10*, 1.3.15*, 1.4.1*, 1.4.11, 1.4.14
*The exercises marked with an asterisk are *not* suggested by Yunied, but in my opinion they may be interesting anyway.

1.3.4. If the sample space if $\mathcal{C} = C_1 \cup C_2$ and if $P(C_1) = 0.8$ and $P(C_2) = 0.5$, find $P(C_1 \cap C_2)$.

Solution. We recall Theorem 1.3.5 on page 12 of the textbook, which asserts that for any events C_1 and C_2, we have

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

Therefore, we conclude

$$\begin{aligned} P(C_1 \cap C_2) &= P(C_1) + P(C_2) - P(C_1 \cup C_2) \\ &= P(C_1) + P(C_2) - P(\mathcal{C}) \\ &= 0.8 + 0.5 - 1 \\ &= 0.3, \end{aligned}$$

as desired. □

1.3.5. Let the sample space be $\mathcal{C} = \{c : 0 < c < \infty\}$. Let $C \subset \mathcal{C}$ be defined by $C = \{c : 4 < c < \infty\}$ and take $P(C) = \int_C e^{-x}\, dx$. Show that $P(\mathcal{C}) = 1$. Evaluate $P(C)$, $P(C^c)$, and $P(C \cup C^c)$.

Solution. The probability of the sample space $\mathcal{C}$ is

$$\begin{aligned} P(\mathcal{C}) &= \int_{\mathcal{C}} e^{-x}\, dx \\ &= \int_0^\infty e^{-x}\, dx \\ &= \lim_{a\to\infty} -e^{-x}|_0^a \\ &= (\lim_{a\to\infty}(-e^{-a})) - (-e^0) \\ &= 0 - (-1) \\ &= 1. \end{aligned}$$

The probability of event C is

$$\begin{aligned} P(C) &= \int_C e^{-x}\, dx \\ &= \int_4^\infty e^{-x}\, dx \\ &= \lim_{a\to\infty} -e^{-x}|_4^a \\ &= (\lim_{a\to\infty}(-e^{-a})) - (-e^4) \\ &= 0 - (-e^4) \\ &= e^4. \end{aligned}$$

Since we have $C = (4, \infty)$, we must have $C^c = \mathcal{C} \setminus C = [0, 4]$. So the probability of event C^c is

$$\begin{aligned} P(C^c) &= \int_C e^{-x}\, dx \\ &= \int_0^4 e^{-x}\, dx \\ &= -e^{-x}|_0^4 \\ &= (-e^{-4}) - (-e^{-0}) \\ &= -e^{-4} - (-1) \\ &= 1 - e^{-4}. \end{aligned}$$

Finally, the probability of event $C \cup C^c$ is

$$\begin{aligned} P(C \cup C^c) &= P(C) + P(C^c) \\ &= e^4 + (1 - e^{-4}) \\ &= 1, \end{aligned}$$

which coincides with the probability of the entire sample space C. □

1.3.10.* A bowl contains 16 chips, of which 6 are red, 7 are white, and 3 are blue. If four chips are taken at random and without replacement, find the probability that:

(a) each of the four chips is red;

Solution. We recall in general that the binomial coefficient $\binom{n}{k}$ represents choosing k objects out of the n available objects. For taking 4 chips out of the 16 chips in the bowl, there are $\binom{16}{4} = \frac{16!}{4!(16-4)!} = 1820$ possibilities. For taking 4 red chips out of the 6 red chips in the bowl, there are $\binom{6}{4} = \frac{6!}{4!(6-4)!} = 15$ possibilities. So we have

$$\begin{aligned} P(\text{all 4 red chips}) &= \frac{15}{1820} \\ &= \frac{3}{364}, \end{aligned}$$

which is the probability that each of the four chips taken at random and without replacement is red. □

(b) none of the four chips is red;

Solution. In this context, saying that none of the four chips is red is equivalent to saying that all four of the chips are either white or blue. As in the solution to part (a), for taking 4 chips out of the 16 chips in the bowl, there are 1820 possibilities. There are 7 white and 3 blue chips in the bowl, which is equivalent to saying there are $7 + 3 = 10$ white or blue chips in the bowl. For taking 4 white or blue chips out of the 10 white or blue chips in the bowl, there are $\binom{10}{4} = \frac{10!}{4!(10-4)!} = 210$ possibilities. So we have

$$\begin{aligned} P(\text{all 4 white or blue chips}) &= \frac{210}{1820} \\ &= \frac{3}{26}, \end{aligned}$$

which is the probability that none of the four chips taken at random and without replacement is red. □

(c) there is at least one chip of each color.

Solution. We need to take four chips, but there are only the three colors of red, white, and blue. This means that, if we require to take at least one chip of each color, we will need one chip of the first color, one chip of the second color, and two chips of the third color. There are three scenarios for which this is possible.

- First, we will work with the case of 2 red chips, 1 white chip, and 1 blue chip.
  - For taking 2 red chips out of the 6 red chips in the bowl, there are $\binom{6}{2} = \frac{6!}{2!(6-2)!} = 15$ possibilities.
  - For taking 1 white chip out of the 7 white chips, there are $\binom{7}{1} = \frac{7!}{1!(7-1)!} = 7$ possibilities.
  - For taking 1 blue chip out of the 3 blue chips, there are $\binom{3}{1} = \frac{3!}{1!(3-1)!} = 3$ possibilities.

  So there are $(15)(7)(3) = 315$ possibilities of taking taking 2 red chips, 1 white chip, and 1 blue chip without replacement, which means that the probability of this occurring is

$$\begin{aligned} P(\text{2 red chips, 1 white chip, 1 blue chip}) &= \frac{315}{1820} \\ &= \frac{9}{52}. \end{aligned}$$

- Next, we will work with the case of 1 red chip, 2 white chips, and 1 blue chip.
  - For taking 1 red chip out of the 6 red chips in the bowl, there are $\binom{6}{1} = \frac{6!}{1!(6-1)!} = 6$ possibilities.
  - For taking 2 white chips out of the 7 white chips, there are $\binom{7}{2} = \frac{7!}{2!(7-2)!} = 21$ possibilities.
  - For taking 1 blue chip out of the 3 blue chips, there are $\binom{3}{1} = \frac{3!}{1!(3-1)!} = 3$ possibilities.

  So there are $(6)(21)(3) = 378$ possibilities of taking taking 1 red chip, 2 white chips, and 1 blue chip without replacement, which means that the probability of this occurring is

$$\begin{aligned} P(\text{1 red chip, 2 white chips, 1 blue chip}) &= \frac{378}{1820} \\ &= \frac{27}{130}. \end{aligned}$$

- Finally, we will work with the case of 1 red chip, 1 white chip, and 2 blue chips.
  - For taking 1 red chip out of the 6 red chips in the bowl, there are $\binom{6}{1} = \frac{6!}{1!(6-1)!} = 6$ possibilities.
  - For taking 1 white chip out of the 7 white chips, there are $\binom{7}{1} = \frac{7!}{1!(7-1)!} = 7$ possibilities.
  - For taking 2 blue chips out of the 3 blue chips, there are $\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3$ possibilities.

  So there are $(6)(7)(3) = 126$ possibilities of taking taking 1 red chip, 1 white chip, and 2 blue chips without replacement, which means that the probability of this occurring is

$$\begin{aligned} P(\text{1 red chip, 1 white chip, 2 blue chips}) &= \frac{126}{1820} \\ &= \frac{9}{130}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} P(\text{at least one chip of each color}) &= P(\text{2 red chips, 1 white chip, 1 blue chip}) \\ &\quad + P(\text{1 red chip, 2 white chips, 1 blue chip}) \\ &\quad + P(\text{1 red chip, 1 white chip, 2 blue chips}) \\ &= \frac{9}{52} + \frac{27}{130} + \frac{9}{130} \\ &= \frac{9}{20}, \end{aligned}$$

which is the probability that there is at least one chip of each color in the four chips taken at random and without replacement. □

1.3.15.* In a lot of 50 light bulbs, there are 2 bad bulbs. An inspector examines five bulbs, which are selected at random and without replacement.

(a) Find the probability of at least one defective bulb among the five.

Solution. For taking 5 light bulbs out of the 50 bulbs in the lot, there are $\binom{50}{5} = \frac{50!}{5!(50-5)!} = 2118760$ possibilities. There are 2 defective bulbs by assumption, which means there are 48 working bulbs. For taking 5 light bulbs out of the 48 working bulbs, there are $\binom{48}{5} = \frac{48!}{5!(48-5)!} = 1712304$ possibilities. Therefore, the probability of having zero defective bulbs among the five is

$$\begin{aligned} P(\text{no defective bulbs}) &= \frac{1712304}{2118760} \\ &= \frac{198}{245}, \end{aligned}$$

which means that

$$\begin{aligned} P(\text{at least one defective bulb}) &= 1 - P(\text{no defective bulbs}) \\ &= 1 - \frac{198}{245} \\ &= \frac{47}{245}. \end{aligned}$$

is the probability of having at least one defective bulb among the five. □

(b) How many bulbs should be examined so that the probability of finding at least one bad bulb exceeds $\frac{1}{2}$?

Solution. For taking n light bulbs out of the 50 bulbs in the lot, there are $\binom{50}{n} = \frac{50!}{n!(50-n)!}$ possibilities. There are 2 defective bulbs by assumption, which means there are 48 working bulbs. For taking n light bulbs out of the 48 working bulbs, there are $\binom{48}{n} = \frac{48!}{n!(48-n)!}$ possibilities. Therefore, the probability of having zero defective bulbs among the n bulbs taken from the lot is

$$\begin{aligned} P(\text{no defective bulbs}) &= \frac{\binom{48}{n}}{\binom{50}{n}} \\ &= \frac{\frac{48!}{n!(48-n)!}}{\frac{50!}{n!(50-n)!}} \\ &= \frac{48!}{n!(48-n)!} \frac{n!(50-n)!}{50!} \\ &= \frac{48!}{50!} \frac{(50-n)!}{(48-n)!} \\ &= \frac{48!}{(50)(49)48!} \frac{(50-n)(49-n)(48-n)!}{(48-n)!} \\ &= \frac{(50-n)(49-n)}{(50)(49)}. \end{aligned}$$

So the probability of finding at least one defective bulb among the n bulbs taken from the lot is

$$\begin{aligned} P(\text{at least one defective bulb}) &= 1 - P(\text{no defective bulbs}) \\ &= 1 - \frac{(50-n)(49-n)}{(50)(49)}. \end{aligned}$$

The problem statement requires that the probability of finding at least one bad bulb exceed $\frac{1}{2}$, which means we require

$$P(\text{at least one defective bulb}) > \frac{1}{2}.$$

Therefore, we have the inequality

$$1 - \frac{(50-n)(49-n)}{(50)(49)} > \frac{1}{2},$$

from which we can solve algebraically to obtain the interval of solutions

$$14.4964 \approx \frac{99 - 13\sqrt{29}}{2} < n < \frac{99 + 13\sqrt{29}}{2} \approx 84.5036.$$

The upper bound of n makes no sense in this context because we can only take up to 50 light bulbs from the 50 total in the lot. So we go with the lower bound $n > \frac{99-13\sqrt{29}}{2} \approx 14.4964$. The smallest integer for this lower bound is 15, meaning that we must take at least 15 light bulbs in order to meet the criterion that the probability of finding at least one bad bulb exceeds $\frac{1}{2}$. □

1.4.1.* If $P(C_1) > 0$ and if $C_2, C_3, C_4, \ldots$ are mutually disjoint sets, show that

$$P(C_2 \cup C_3 \cup \cdots | C_1) = P(C_2|C_1) + P(C_3|C_1) + \cdots.$$

Proof. By the definition of the definition of conditional probability, we have

$$P(C_2 \cup C_3 \cup \cdots | C_1) = \frac{P(C_1 \cap (C_2 \cup C_3 \cup \cdots))}{P(C_1)}.$$

Because set intersections distribute over set unions, we can write

$$C_1 \cap (C_2 \cup C_3 \cup \cdots) = (C_1 \cap C_2) \cup (C_1 \cap C_3) \cup \cdots.$$

Finally, since we are given that $C_2, C_3, C_4, \ldots$ are mutually disjoint, it follows that $C_1 \cap C_2, C_1 \cap C_3, C_1 \cap C_4, \ldots$ are also mutually disjoint; indeed, by the distributive property of set intersections, we observe

$$\begin{aligned} (C_1 \cap C_2) \cap (C_1 \cap C_3) \cap (C_1 \cap C_4) \cap \cdots &= C_1 \cap (C_2 \cap C_3 \cap C_4 \cap \cdots) \\ &= C_1 \cap \varnothing \\ &= \varnothing. \end{aligned}$$

So we can invoke the third axiom of probability to write

$$P((C_1 \cap C_2) \cup (C_1 \cap C_3) \cup \cdots) = P(C_1 \cap C_2) + P(C_1 \cap C_3) + \cdots.$$

Putting our results together, we conclude

$$\begin{aligned} P(C_2 \cup C_3 \cup \cdots | C_1) &= \frac{P(C_1 \cap (C_2 \cup C_3 \cup \cdots))}{P(C_1)} \\ &= \frac{P((C_1 \cap C_2) \cup (C_1 \cap C_3) \cup \cdots)}{P(C_1)} \\ &= \frac{P(C_1 \cap C_2) + P(C_1 \cap C_3) + \cdots}{P(C_1)} \\ &= \frac{P(C_1 \cap C_2)}{P(C_1)} + \frac{P(C_1 \cap C_3)}{P(C_1)} + \cdots \\ &= P(C_2|C_1) + P(C_3|C_1) + \cdots, \end{aligned}$$

as desired. □

1.4.11. If C_1 and C_2 are independent events, show that the following pairs of events are also independent:

(a) C_1 and C_2^c
Hint: Write $P(C_1 \cap C_2^c) = P(C_1)P(C_2^c|C_1) = P(C_1)(1 - P(C_2|C_1))$. From the independence of C_1 and C_2, we have $P(C_2|C_1) = P(C_2)$.

Proof. Following the given hint and using the definition of conditional probability, we have

$$\begin{aligned} P(C_1|C_2^c) &= \frac{P(C_1 \cap C_2^c)}{P(C_2^c)} \\ &= \frac{P(C_1)P(C_2^c|C_1)}{P(C_2^c)} \\ &= \frac{P(C_1)(1 - P(C_2|C_1)}{1 - P(C_2)} \\ &= \frac{P(C_1)(1 - P(C_2))}{1 - P(C_2)} \\ &= P(C_1), \end{aligned}$$

which means C_1 and C_2^c are independent. □

(b) C_1^c and C_2

Proof. The idea here is to follow the proof of part (a), but interchange the roles of C_1 and C_2; in other words, we will reproduce the proof of part (a), but replace all instances of C_1 with C_2 and C_2 with C_1. Using the definition of conditional probability, we have

$$\begin{aligned} P(C_2|C_1^c) &= \frac{P(C_2 \cap C_1^c)}{P(C_1^c)} \\ &= \frac{P(C_2)P(C_1^c|C_2)}{P(C_1^c)} \\ &= \frac{P(C_2)(1 - P(C_1|C_2)}{1 - P(C_1)} \\ &= \frac{P(C_2)(1 - P(C_1))}{1 - P(C_1)} \\ &= P(C_2), \end{aligned}$$

which means C_2 and C_1^c are independent, which, in turn, is obviously equivalent to saying C_1^c and C_2 are independent. □

(c) C_1^c and C_2^c

Proof. One of De Morgan's Laws (see Example 1.2.17 on page 6 of the textbook) asserts that, if C_1 and C_2 are sets, then we have

$$(C_1 \cup C_2)^c = C_1^c \cap C_2^c.$$

We also recall Theorem 1.3.5 on page 12 of the textbook, which asserts that for any events C_1 and C_2, we have

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

Finally, since we assumed in the hypothesis that C_1 and C_2 are independent events, we have $P(C_2|C_1) = P(C_2)$, and so the definition of conditional probability implies

$$\begin{aligned} P(C_1 \cap C_2) &= P(C_1)P(C_2|C_1) \\ &= P(C_1)P(C_2). \end{aligned}$$

Applying all of the above and the definition of conditional probability, we have

$$\begin{aligned}
P(C_2^c|C_1^c) &= \frac{P(C_2^c \cap C_1^c)}{P(C_1^c)} \\
&= \frac{P((C_1 \cup C_2)^c)}{P(C_1^c)} \\
&= \frac{1 - P(C_1 \cup C_2)}{P(C_1^c)} \\
&= \frac{1 - (P(C_1) + P(C_2) - P(C_1 \cap C_2))}{P(C_1^c)} \\
&= \frac{1 - P(C_1) - P(C_2) + P(C_1 \cap C_2)}{P(C_1^c)} \\
&= \frac{1 - P(C_1) - P(C_2) + P(C_1)P(C_2)}{P(C_1^c)} \\
&= \frac{(1 - P(C_1))(1 - P(C_2))}{P(C_1^c)} \\
&= \frac{P(C_1^c)P(C_2^c)}{P(C_1^c)} \\
&= P(C_2^c),
\end{aligned}$$

which means C_1^c and C_2^c are independent. □

1.4.14. Each of four persons fires one shot at a target. Let C_k denote the event that the target is hit by person k, $k = 1, 2, 3, 4$. If C_1, C_2, C_3, C_4 are independent and if $P(C_1) = P(C_2) = 0.7$, $P(C_3) = 0.9$, and $P(C_4) = 0.4$, compute the probability that

(a) all of them hit the target;

Solution. We have

$$\begin{aligned}
P(\text{all of } C_1, C_2, C_3, C_4 \text{ hit}) &= P(C_1 \cap C_2 \cap C_3 \cap C_4) \\
&= P(C_1)P(C_2)P(C_3)P(C_4) \\
&= (0.7)(0.7)(0.9)(0.4) \\
&= 0.1764,
\end{aligned}$$

meaning that there is a 17.64% chance that all of them hit the target. □

(b) exactly one hits the target;

Solution. The probability that only person 1 hits the target is

$$\begin{aligned}
P(C_1 \text{ hits and } C_2, C_3, C_4 \text{ miss}) &= P(C_1 \cap C_2^c \cap C_3^c \cap C_4^c) \\
&= P(C_1)P(C_2^c)P(C_3^c)P(C_4^c) \\
&= P(C_1)(1 - P(C_2))(1 - P(C_3))(1 - P(C_4)) \\
&= (0.7)(1 - 0.7)(1 - 0.9)(1 - 0.4) \\
&= 0.0126.
\end{aligned}$$

The probability that only person 2 hits the target is

$$\begin{aligned}
P(C_2 \text{ hits and } C_1, C_3, C_4 \text{ miss}) &= P(C_1^c \cap C_2 \cap C_3^c \cap C_4^c) \\
&= P(C_1^c)P(C_2)P(C_3^c)P(C_4^c) \\
&= (1 - P(C_1))P(C_2)(1 - P(C_3))(1 - P(C_4)) \\
&= (1 - 0.7)(0.7)(1 - 0.9)(1 - 0.4) \\
&= 0.0126.
\end{aligned}$$

The probability that only person 3 hits the target is

$$\begin{aligned}
P(C_3 \text{ hits and } C_1, C_2, C_4 \text{ miss}) &= P(C_1^c \cap C_2^c \cap C_3 \cap C_4^c) \\
&= P(C_1^c)P(C_2^c)P(C_3)P(C_4^c) \\
&= (1 - P(C_1))(1 - P(C_2))P(C_3)(1 - P(C_4)) \\
&= (1 - 0.7)(1 - 0.7)(0.9)(1 - 0.4) \\
&= 0.0486.
\end{aligned}$$

The probability that only person 4 hits the target is

$$\begin{aligned}
P(C_4 \text{ hits and } C_1, C_2, C_3 \text{ miss}) &= P(C_1^c \cap C_2^c \cap C_3^c \cap C_4) \\
&= P(C_1^c)P(C_2^c)P(C_3^c)P(C_4) \\
&= (1 - P(C_1))(1 - P(C_2))(1 - P(C_3))P(C_4) \\
&= (1 - 0.7)(1 - 0.7)(1 - 0.9)(0.4) \\
&= 0.0036.
\end{aligned}$$

Therefore, the probability that exactly one person out of any of these four people hit the target is

$$\begin{aligned}
P(\text{exactly one of } C_1, C_2, C_3, C_4 \text{ hits}) &= P(C_1 \text{ hits and } C_2, C_3, C_4 \text{ miss}) + P(C_2 \text{ hits and } C_1, C_3, C_4 \text{ miss}) \\
&\quad + P(C_3 \text{ hits and } C_1, C_2, C_4 \text{ miss}) + P(C_4 \text{ hits and } C_1, C_2, C_3 \text{ miss}) \\
&= 0.0126 + 0.0126 + 0.0486 + 0.0036 \\
&= 0.0774,
\end{aligned}$$

meaning that there is a 7.74% chance that exactly one hits the target. □

(c) no one hits the target;

Solution. Saying that no one hits the target is equivalent to saying that all of them miss the target. Also, since C_1, C_2, C_3, C_4 are independent events, it follows by an application of Exercise 1.4.11(c) that $C_1^c, C_2^c, C_3^c, C_4^c$ are also independent events. So we have

$$\begin{aligned}
P(\text{all of } C_1, C_2, C_3, C_4 \text{ miss}) &= P(C_1^c \cap C_2^c \cap C_3^c \cap C_4^c) \\
&= P(C_1^c)P(C_2^c)P(C_3^c)P(C_4^c) \\
&= (1 - P(C_1))(1 - P(C_2))(1 - P(C_3))(1 - P(C_4)) \\
&= (1 - 0.7)(1 - 0.7)(1 - 0.9)(1 - 0.4) \\
&= 0.0054,
\end{aligned}$$

meaning that there is a 0.54% chance that no one hits the target. □

(d) at least one hits the target.

Solution. Observe that the probability that at least one hits the target is the compliment of the probability that no one hits the target. In other words, we have

$$\begin{aligned}
P(\text{at least one of } C_1, C_2, C_3, C_4 \text{ hits}) &= 1 - P(\text{all of } C_1, C_2, C_3, C_4 \text{ miss}) \\
&= 1 - 0.0054 \\
&= 0.9946,
\end{aligned}$$

meaning that there is a 99.46% chance that at least one hits the target. □