Theorem 1.8.2

Note: This is Theorem 1.8.2 (page 55) of the textbook *Introduction to Mathematical Statistics* (seventh edition) by Robert V. Hogg, Joseph W. McKean, Allen T. Craig. I am following the proof of Theorem 1.8.2 but filling in intermediate steps here, so that the proof is hopefully easier to read.

Attention: In discussion, I only proved the existence of the expectation of $k_1 g_1(X) + k_2 g_2(X)$ in both the continuous and discrete cases. However, I forgot to address in discussion how to establish $E(k_1 g_1(X) + k_2 g_2(X)) = k_1 E(g_1(X)) + k_2 E(g_2(X))$. But this equality is addressed in the proof below for both the continuous and discrete cases. Please make sure you know this as well for your midterm!

Theorem. *Let $g_1(X)$ and $g_2(X)$ be functions of a random variable X. Suppose the expectations of $g_1(X)$ and $g_2(X)$ exist. Then, for any constants k_1 and k_2, the expectation of $k_1 g_1(X) + k_2 g_2(X)$ exists and it is given by*

$$E(k_1 g_1(X) + k_2 g_2(X)) = k_1 E(g_1(X)) + k_2 E(g_2(X)).$$

Proof. First, we will work in the *continuous case*. Since we assumed that the expectations of $g_1(X)$ and $g_2(X)$ exist, we automatically have

$$|E(g_1(X))| \leq \int_{-\infty}^{\infty} |g_1(x)| f_X(x)\, dx < \infty,$$
$$|E(g_2(X))| \leq \int_{-\infty}^{\infty} |g_2(x)| f_X(x)\, dx < \infty.$$

Therefore, using the triangle inequality (second step below) linearity of the integral (fourth step below), we have

$$\begin{aligned}
|E(k_1 g_1(X) + k_2 g_2(X))| &\leq \int_{-\infty}^{\infty} |k_1 g_1(x) + k_2 g_2(x)| f_X(x)\, dx \\
&\leq \int_{-\infty}^{\infty} (|k_1 g_1(x)| + |k_2 g_2(x)|) f_X(x)\, dx \\
&\leq \int_{-\infty}^{\infty} |k_1||g_1(x)| f_X(x) + |k_2||g_2(x)| f_X(x)\, dx \\
&= |k_1| \int_{-\infty}^{\infty} |g_1(X)| f_X(x)\, dx + |k_2| \int_{-\infty}^{\infty} |g_2(x)| f_X(x)\, dx \\
&< \infty,
\end{aligned}$$

which means the expectation of $k_1 g_1(X) + k_2 g_2(X)$ exists. Finally, we use the linearity of the integral to establish

$$\begin{aligned}
E(k_1 g_1(X) + k_2 g_2(X)) &= \int_{-\infty}^{\infty} (k_1 g_1(x) + k_2 g_2(x)) f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} k_1 g_1(x) f_X(x) + k_2 g_2(x) f_X(x)\, dx \\
&= k_1 \int_{-\infty}^{\infty} g(x) f_X(x)\, dx + k_2 \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \\
&= k_1 E(g_1(X)) + k_2 E(g_2(X)),
\end{aligned}$$

as desired.

Next, we will work in the *discrete case*. Since we assumed that the expectations of $g_1(X)$ and $g_2(X)$ exist, we automatically have

$$|E(g_1(X))| \leq \sum_x |g_1(x)| p_X(x) < \infty,$$
$$|E(g_2(X))| \leq \sum_x |g_2(x)| p_X(x) < \infty.$$

Therefore, using the triangle inequality (second step below) linearity of the sum (fourth step below), we have

$$\begin{aligned}
|E(k_1 g_1(X) + k_2 g_2(X))| &\leq \sum_x |k_1 g_1(x) + k_2 g_2(x)| p_X(x) \\
&\leq \sum_x (|k_1 g_1(x)| + |k_2 g_2(x)|) p_X(x) \\
&\leq \sum_x (|k_1||g_1(x)| p_X(x) + |k_2||g_2(x)| p_X(x)) \\
&= |k_1| \sum_x |g_1(x)| p_X(x) + |k_2| \sum_x |g_2(x)| p_X(x) \\
&< \infty,
\end{aligned}$$

which means the expectation of $k_1 g_1(X) + k_2 g_2(X)$ exists. Finally, we use the linearity of the sum to establish

$$\begin{aligned} E(k_1 g_1(X) + k_2 g_2(X)) &= \sum_x (k_1 g_1(x) + k_2 g_2(x)) p_X(x) \\ &= \sum_x k_1 g_1(x) p_X(x) + k_2 g_2(x) p_X(x) \\ &= k_1 \sum_x g(x) p_X(x) + k_2 \sum_x g(x) p_X(x) \\ &= k_1 E(g_1(X)) + k_2 E(g_2(X)), \end{aligned}$$

as desired. □