MATH 149A discussion — Ryan Ta
University of California, Riverside — Fall 2019

Notes for Week 5 discussion on October 29

Let f_X be a probability density function (pdf), which is nonnegative. An *expectation* of a random variable X exists if we have

$$\int_{-\infty}^{\infty} |x| f_X(x)\, dx < \infty.$$

Note that the above equality is a sufficient condition for existence; a necessary condition (think "necessary and sufficent" if you took a logic course!) is

$$|E(X)| < \infty.$$

Indeed, assuming that the expectation of X exists, we have

$$\begin{aligned} |E(X)| &= \left| \int_{-\infty}^{\infty} x f_X(x)\, dx \right| \\ &\leq \int_{-\infty}^{\infty} |x||f_X(x)|\, dx \\ &\leq \int_{-\infty}^{\infty} |x| f_X(x)\, dx \\ &< \infty. \end{aligned}$$

The next item we addressed in our discussion class is Theorem 1.8.2, which is most likely on your upcoming midterm. I placed that on a separate document, accessible from our discussion webpage, so we will skip that in this document here. The rest of this document will address a few textbook examples that we went over in discussion. In these next examples, keep in mind whether we are in the continuous or discrete case!

Example (Example 1.8.4 of the textbook)**.** *Let X have the pdf*

$$f_X(x) = \begin{cases} 2(1-x) & 0 < x < 1, \\ 0 & \textit{otherwise}. \end{cases}$$

Then we have

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\ &= \int_0^1 x(2(1-x))\, dx \\ &= 2\int_0^1 x - x^2\, dx \\ &= \frac{1}{3} \end{aligned}$$

and

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \\ &= \int_0^1 x^2(2(1-x))\, dx \\ &= 2\int_0^1 x^2 - x^3\, dx \\ &= \frac{1}{6}. \end{aligned}$$

We can also use linearity of the expectation, which we established in Theorem 1.8.2, to obtain, for instance,

$$\begin{aligned} E(3X^2 + 6X) &= 3E(X^2) + 6E(X) \\ &= 3 \cdot \frac{1}{6} + 6 \cdot \frac{1}{3} \\ &= \frac{1}{2}. \end{aligned}$$

Example (Example 1.8.5 of the textbook)**.** *Let X have the pmf*

$$p_X(x) = \begin{cases} \frac{x}{6} & x = 1, 2, 3, \\ 0 & \textit{otherwise}. \end{cases}$$

Then we have

$$\begin{aligned} E(X) &= \sum_x x p_X(x) \\ &= 1p_X(1) + 2p_X(2) + 3p_X(3) \\ &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{3}{6} \\ &= \frac{7}{3} \end{aligned}$$

and

$$\begin{aligned} E(X^3) &= \sum_x x^3 p_X(x) \\ &= 1p_X(1) + 2p_X(2) + 3p_X(3) \\ &= 1^3 \cdot \frac{1}{6} + 2^3 \cdot \frac{2}{6} + 3^3 \cdot \frac{3}{6} \\ &= \frac{49}{3} \end{aligned}$$

We can also use linearity of the expectation, which we established in Theorem 1.8.2, to obtain, for instance,

$$\begin{aligned} E(2X^3 + 3X) &= 2E(X^3) + 3E(X) \\ &= 2 \cdot \frac{49}{3} + 3 \cdot \frac{7}{3} \\ &= \frac{119}{3}. \end{aligned}$$