Notes for Week 6 discussion on November 5

**Definition.** *Let $X_1$ and $X_2$ be two random variables associated with some experiment with sample space c. This means we would have $X_1(c) = x_1$ and $X_2(c) = x_2$. We say that $(X_1, X_2)$ is a* random vector. *Furthermore, the* space of $(X_1, X_2)$ *is the set of ordered paris $\mathcal{D} = \{(x_1, x_2) \in \mathbb{R}^2 : X_1(c) = x_1, X_2(c) = x_2, c \in C\}$.*

**Definition.** *The* cumulative distribution function *(cdf) is given by*

$$F_{X_1,X_2}(x_1, x_2) = P((X_1 \le x_1) \cap (X_2 \le x_2)).$$

*(Chapter 1 analogue: $F_X(x) = P(X \le x)$.)*

**Definition.** *The* joint probability mass function *(pmf) is given by*

$$p_{X_1,X_2}(x_1, x_2) = P(X_1 = x_1, X_2 \le x_2).$$

*(Chapter 1 analogue: $p_X(x) = P(X \le x)$.)*

**Definition.** *The* joint probability mass function *(pdf) is given by*

$$f_{X_1,X_2}(x_1, x_2) = \frac{\partial^2 F_{X_1,X_2}(x_1, x_2)}{\partial x_1 \partial x_2}.$$

*(Chapter 1 analogue: $f_X(x) = \frac{dF(x)}{dx}$.)*

**Example** (Example 2.1.2 of the textbook)**.** *Let*

$$f(x_1, x_2) := \begin{cases} 6x_1^2 x_2 & \text{if } 0 < x_1 < 1, 0 < x_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

*(a) Show $P(0 < x_1 < 1, 0 < x_2 < 1) = 1$.*

*Solution.* We have

$$\begin{aligned} P(0 < x_1 < 1, 0 < x_2 < 1) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1\, dx_2 \\ &= \int_0^1 \int_0^1 6x_1^2 x_2\, dx_1\, dx_2 \\ &= \int_0^1 2x_1^3|_0^1 x_2\, dx_2 \\ &= \int_0^1 2(1^3 - 0^3) x_2\, dx_2 \\ &= \int_0^1 2x_2\, dx_2 \\ &= x_2^2|_0^1 \\ &= (1^2 - 0^2) \\ &= 1, \end{aligned}$$

as desired. □

*(b) Compute $P(0 < X_1 < \frac{3}{4}, \frac{1}{2} < X < 2)$.*

*Proof.* We have

$$\begin{aligned}
P\left(0 < X_1 < \frac{3}{4}, \frac{1}{3} < X_2 < 2\right) &= \int_{\frac{1}{3}}^{2} \int_{0}^{\frac{3}{4}} f(x_1, x_2)\, dx_1\, dx_2 \\
&= \int_{\frac{1}{3}}^{1} \int_{0}^{\frac{3}{4}} f(x_1, x_2)\, dx_1\, dx_2 + \int_{1}^{2} \int_{0}^{\frac{3}{4}} f(x_1, x_2)\, dx_1\, dx_2 \\
&= \int_{\frac{1}{3}}^{1} \int_{0}^{\frac{3}{4}} 6x_1^2 x_2\, dx_1\, dx_2 + \int_{1}^{2} \int_{0}^{\frac{3}{4}} 0\, dx_1\, dx_2 \\
&= \int_{\frac{1}{3}}^{1} \int_{0}^{\frac{3}{4}} 6x_1^2 x_2\, dx_1\, dx_2 \\
&= \int_{\frac{1}{3}}^{1} 2x_1^3|_0^{\frac{3}{4}} x_2\, dx_2 \\
&= \int_{\frac{1}{3}}^{1} 2\left(\left(\frac{3}{4}\right)^3 - (0)^3\right) x_2\, dx_2 \\
&= \frac{27}{32} \int_{\frac{1}{3}}^{1} x_2\, dx_2 \\
&= \frac{27}{32} \frac{1}{2} x_2^2|_{\frac{1}{3}}^{1} \\
&= \frac{27}{74} \left((1)^2 - \left(\frac{1}{3}\right)^2\right) \\
&= \frac{3}{8},
\end{aligned}$$

as desired. □

(c) *Find the cdf of $X_1, X_2$.*

*Proof.* We have

$$\begin{aligned}
F_{X_1,X_2}(x_1, x_2) &= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{X_1,X_2}(w_1, w_2)\, dw_1\, dw_2 \\
&= \int_{0}^{x_1} \int_{0}^{x_2} 6w_1^2 w_2\, dw_1\, dw_2 \\
&= \int_{0}^{x_1} 2w_1^3|_0^{x_2} w_2\, dw_2 \\
&= \int_{0}^{x_1} 2(x_2^3 - 0^3) w_1\, dw_2 \\
&= 2x_2^3 \int_{0}^{x_1} 2w_1\, dw_2 \\
&= x_2^3 w_1^2|_0^{x_1} \\
&= x_2^3(x_1^2 - 0^2) \\
&= x_1^2 x_2^3,
\end{aligned}$$

as desired. □

**Example** (Example 2.1.5 of the textbook)**.** *Let*

$$f(x_1, x_2) := \begin{cases} 8x_1x_2 & \textit{if } 0 < x_1 < x_2 < 1, \\ 0 & \textit{otherwise.} \end{cases}$$

*Find $E(X_1X_2^2)$, $E(X_2)$, and $E(7X_1X_2^2 + 5X_2)$.*

*Solution.* We have

$$\begin{aligned}
E(X_1X_2^2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2^2 f(x_1,x_2)\,dx_2\,dx_1 \\
&= \int_0^1\int_{x_1}^1 x_1x_2^2(8x_1x_2)\,dx_2\,dx_1 \\
&= 8\int_0^1\int_{x_1}^1 x_1^2x_2^3\,dx_2\,dx_1 \\
&= 2\int_0^1 x_2x_2^4|_{x_1}^1\,dx_1 \\
&= 2\int_0^1 x_1^2(1^4-x_1^4)\,dx_1 \\
&= 2\int_0^1 x_1^2-x_1^6\,dx_1 \\
&= 2\left(\frac{x_1^3}{3}-\frac{x_1^7}{7}\right)\Bigg|_0^1 \\
&= 2\left(\left(\frac{(1)^3}{3}-\frac{(1)^7}{7}\right)-\left(\frac{(0)^3}{3}-\frac{(0)^7}{7}\right)\right) \\
&= \frac{8}{21},
\end{aligned}$$

as desired. We also have

$$\begin{aligned}
E(X_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 f(x_1,x_2)\,dx_2\,dx_1 \\
&= \int_0^1\int_{x_1}^1 x_2(8x_1x_2)\,dx_2\,dx_1 \\
&= 8\int_0^1 x_1\int_{x_1}^1 x_2^2\,dx_2\,dx_1 \\
&= 8\int_0^1 x_1\left.\frac{x_2^3}{3}\right|_{x_1}^1\,dx_1 \\
&= 8\int_0^1 x_1\left(\frac{(1)^3}{3}-\frac{x_1^3}{3}\right)\,dx_1 \\
&= \frac{8}{3}\int_0^1 x_1-x_1^4\,dx_1 \\
&= \frac{8}{3}\left(\frac{x_1^2}{2}-\frac{x_1^5}{5}\right)\Bigg|_0^1 \\
&= \frac{8}{3}\left(\left(\frac{(1)^2}{2}-\frac{(1)^5}{5}\right)-\left(\frac{(0)^2}{2}-\frac{(0)^5}{5}\right)\right) \\
&= \frac{4}{5}.
\end{aligned}$$

Consequently, using the linearity of expectation, we have

$$\begin{aligned}
E(7X_1X_2^2+5X_2) &= 7E(X_1X_2^2)+5E(X_2) \\
&= 7\left(\frac{8}{21}\right)+5\left(\frac{4}{5}\right) \\
&= \frac{20}{3},
\end{aligned}$$

as desired. □