MATH 149A discussion
University of California, Riverside

Ryan Ta
Fall 2019

Notes for Week 7 discussion on November 12

For the examples of today's notes, we will list below two important inequalities. We will not prove these inequalities in this document. Please see the PDF for November 19 to find their proofs.

Theorem (Markov's inequality)**.** *Let $u(X)$ be a nonnegative function of the random variable X. If $E[u(X)]$ exists, then for every constant $c > 0$, we have*

$$P[u(X) \geq c] \leq \frac{E[u(X)]}{c}.$$

Theorem (Chebyshev's inequality)**.** *Let X be the random variable with finite variance σ^2 (which implies that the mean $\mu = E(X)$ exists). Then, for all $k \geq 0$, we have*

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

The examples below will make use of Markov's Inequality and Chebyshev's Inequality, which is why we mentioned their statements above in this document. Let X be a random variable, let $\mu = E(X)$ denote the mean, and let $\sigma^2 = \text{Var}(X) = E(X^2) - (E(X))^2$.

Example. *Let X be a random variable such that $E(X-\mu)^2$ exists. Show that, for all $c > 0$, we have*

$$P(|X - \mu| > c) = P(|X - \mu|^2 > c^2).$$

Proof. Since $|X - \mu| > c$ is equivalent to $(X - \mu)^2 = |X - \mu|^2 > c^2$, we have

$$P(|X - \mu| > c) - P((X - \mu)^2 > c^2).$$

Also, we observe $((X - \mu)^2 > c^2) \subseteq ((X - \mu)^2 \geq c^2)$, we have

$$P((X - \mu)^2 > c^2) \leq P((X - \mu)^2 \geq c^2).$$

Since $(X - \mu)^2$ is nonnegative, we may use Markov's inequality to assert

$$P((X - \mu)^2 \geq c^2) \leq \frac{E((X - \mu)^2)}{c^2}.$$

Putting all our results together, we conclude

$$\begin{aligned} P(|X - \mu| > c) &= P((X - \mu)^2 > c^2) \\ &= P((X - \mu)^2 > c^2) \\ &\leq P((X - \mu)^2 \geq c^2) \\ &\leq \frac{E((X - \mu)^2)}{c^2}, \end{aligned}$$

as desired. □

Example. *Let X be a continuous random variable with its pdf given by*

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(|X - \mu| \geq 2\sigma)$.

Solution. We have

$$\begin{aligned} \mu = E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_0^{\infty} x(2e^{-2x})\, dx \\ &= 2\int_0^{\infty} xe^{-2x}\, dx \\ &= \frac{1}{2}, \end{aligned}$$

which means $2\mu = 1$, and

$$\begin{aligned} E(X^2) &= \int_0^\infty x^2 f_X(x)\,dx \\ &= \int_0^\infty x^2(2e^{-2x})\,dx \\ &= \int_0^\infty 2x^2 e^{-2x}\,dx \\ &= \frac{1}{2}. \end{aligned}$$

So we have

$$\begin{aligned} \sigma^2 = \mathrm{Var}(X) &= E(X^2) - (E(X))^2 \\ &= \frac{1}{2} - \left(\frac{1}{2}\right)^2 \\ &= \frac{1}{4}, \end{aligned}$$

which means $\sigma = \frac{1}{2}$. So we have

$$\begin{aligned} P(|X - 2\mu| > \sigma) &= P\left(|X - 1| > \frac{1}{2}\right) \\ &= P\left(\left(X - \frac{1}{2} \geq 1\right) \cup \left(X - \frac{1}{2} \leq -1\right)\right) \\ &= P\left(X - \frac{1}{2} \geq 1\right) + P\left(X - \frac{1}{2} \leq -1\right) \\ &= P\left(X \geq \frac{3}{2}\right) + P\left(X \leq -\frac{1}{2}\right) \\ &= \int_{\frac{3}{2}}^\infty f_X(x)\,dx + \int_{-\infty}^{-\frac{1}{2}} f_X(x)\,dx \\ &= \int_{\frac{3}{2}}^\infty 2e^{-2x}\,dx + \int_{-\infty}^{-\frac{1}{2}} 0\,dx \\ &= \int_{\frac{3}{2}}^\infty 2e^{-2x}\,dx \\ &= e^{-3}, \end{aligned}$$

as desired. □

Example (Example 1.10.1 of the textbook)**.** *Let X have the pdf*

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{3}} & -\sqrt{3} < x < \sqrt{3}, \\ 0 & \textit{otherwise}. \end{cases}$$

Find $P(|X - \mu| \geq k\sigma)$ if $k = \frac{3}{2}$. Also, use Chebyshev's inequality to obtain an upper bound of $P(|X - \mu| \geq k\sigma)$ if $k = \frac{3}{2}$.

Solution. We have

$$\begin{aligned} \mu = E(X) &= \int_{-\infty}^\infty x f_X(x)\,dx \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x}{2\sqrt{3}}\,dx \\ &= \left.\frac{x^2}{4\sqrt{3}}\right|_{-\sqrt{3}}^{\sqrt{3}} \\ &= \frac{1}{4\sqrt{3}}((\sqrt{3})^2 - (-\sqrt{3})^2) \\ &= 0, \end{aligned}$$

which means $2\mu = 0$, and

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x^2}{2\sqrt{3}}\,dx \\ &= \frac{x^3}{6\sqrt{3}}\bigg|_{-\sqrt{3}}^{\sqrt{3}} \\ &= \frac{1}{6\sqrt{3}}((\sqrt{3})^3 - (-\sqrt{3})^3) \\ &= 1. \end{aligned}$$

So we have

$$\begin{aligned} \sigma^2 = \mathrm{Var}(X) &= E(X^2) - (E(X))^2 \\ &= 1 - 0^2 \\ &= 1, \end{aligned}$$

which means $\sigma = 1$. So we have

$$\begin{aligned} P(|X - \mu| \geq k\sigma) &= P\left(|X| \geq \frac{3}{2}\right) \\ &= 1 - P\left(|X| < \frac{3}{2}\right) \\ &= 1 - \int_{-\infty}^{\frac{3}{2}} f_X(x)\,dx \\ &= \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2\sqrt{3}}\,dx \\ &= \frac{x}{2\sqrt{3}}\bigg|_{-\frac{3}{2}}^{\frac{3}{2}} \\ &= \frac{1}{2\sqrt{3}}\left(\frac{3}{2} - \left(-\frac{3}{2}\right)\right) \\ &= \frac{\sqrt{3}}{2}, \end{aligned}$$

as desired. Now, Chebyshev's inequality gives the upper bound

$$\begin{aligned} P(|X - \mu| \geq k\sigma) &= P\left(|X| \geq \frac{3}{2}\sigma\right) \\ &\leq \frac{1}{(\frac{3}{2})^2} \\ &= \frac{4}{9}, \end{aligned}$$

which indeed satisfies $P(|X - \mu| \geq k\sigma) = \frac{\sqrt{3}}{2} \leq \frac{4}{9}$. □

Example. *Suppose X is a random variable that satisfies $\mu = E(X) = 13$ and $\sigma^2 = \mathrm{Var}(X) = 25$. Use Chebyshev's inequality to give a lower bound of $P(5 < X < 21)$.*

Solution. First, we observe that $5 < X < 21$ is equivalent to $-8 < X - 13 < 8$ (subtract 13 on each term of $5 < X < 21$), which is in turn equivalent to $|X - 13| < 8$. So we have

$$\begin{aligned} P(5 < X < 21) &= P(-8 < X - 13 < 8) \\ &= P(|X - 13| < 8) \\ &= 1 - P(|X - 13| \geq 8). \end{aligned}$$

Now, since we were given $\sigma^2 = 25$, which implies $\sigma = 5$, Chebyshev's inequality asserts

$$\begin{aligned} P(|X - 13| \geq 8) &= P\left(|X - 13| \geq \frac{8}{5}\sigma\right) \\ &\leq \frac{1}{(\frac{8}{5})^2} \\ &= \frac{25}{64}. \end{aligned}$$

Therefore, we conclude

$$\begin{aligned} P(5 < X < 21) &= 1 - P(|X - 13| \geq 8) \\ &\geq 1 - \frac{25}{64} \\ &= \frac{39}{64}, \end{aligned}$$

meaning that $\frac{39}{64}$ is the lower bound of $P(5 < X < 21)$ provided by Chebyshev's inequality. $\square$