Math 150A, Winter 2020–Final Exam

Except for the extra credit problem, this exam has 6 problems worth 60 points distributed over 10 pages, including this one and one blank page. The extra credit problem is at the last page 11. Please read the following instruction very carefully.

Instruction: This is a three-hour open notes exam. Please give yourself 3 hours between 2:50pm and 6:10pm on March 20, 2020. That means, once you open this exam, you have to finish it within 3 hours. You have to email it back to me by 6:10pm. No late submission will be accepted. You may consult your notes or textbook during the exam. But you may not use anything else. In particular, you have to work independently. *Highly similar exams will be reported as cheating cases to the Student Conduct & Academic Integrity Programs.*

Show all of your work on each problem and carefully justify all answers. Points will be deducted for irrelevant, incoherent or incorrect statements, and no points will be awarded for illegible work. If you run out of room, you may work answers on the blank page. Be sure to clearly indicate when work is continued on another page.

NAME (please print): Z. Z.

NetID (please print):

Honor Pledge

On my honor, I have neither given nor received any unauthorized aid on this exam.

Signature:

Question	Points	Score	
1	10		J. A.
2	10		J. A.
3	10		R. T.
4	10		R. T.
5	10		R. T.
6	10		Z. Z.
Total:	60		

Extra Credit Problem Z. Z.

1. (a) (5 points) Consider the sequence $\{a_n = \frac{\cos(n\pi)}{n^2+1}\}_{n\in\mathbb{Z}_+}$. Show by definition that it's convergent.

Guess $0 = \lim\limits_{n\to\infty} a_n$ (1pt)

$\forall \varepsilon > 0$, want to find $n_0 \in \mathbb{Z}_+$ s.t. $|a_n - 0| < \varepsilon$ $\forall n \geq n_0$ (1pt)

Work backwards:

$$|a_n - 0| = \left|\frac{\cos(n\pi)}{n^2+1}\right| = \frac{1}{n^2+1} < \varepsilon$$ (2pts)

$$\Leftarrow \frac{1}{n^2} < \varepsilon \iff n^2 > \frac{1}{\varepsilon} \iff n > \sqrt{\frac{1}{\varepsilon}}$$

Thus by taking $n_0 = \lceil \sqrt{\frac{1}{\varepsilon}} \rceil$, we would have (1pt)

$|a_n - 0| < \varepsilon$ $\forall n \geq n_0$

They must use definition to show convergence.
But they use other method to guess the limit 0.

(b) (5 points) Consider the sequence $\{a_n = (-1)^n \cdot \frac{n}{n+1}\}_{n\in\mathbb{Z}_+}$. Is it divergent or convergent? Explain your reasoning?

$\{a_n\}$ is divergent.

Take subsequence $n = 2m-1$, $m \in \mathbb{Z}_+$, (2pts)

we get $a_{2m-1} = -1 \cdot \frac{2m-1}{2m} \to -1$ as $m \to \infty$

Take another subsequence $n = 2m$, $m \in \mathbb{Z}_+$, (2pts)

we get $a_{2m} = \frac{2m}{2m+1} \to 1$ as $m \to \infty$

Since $\lim\limits_{m\to\infty} a_{2m-1} = -1 \neq 1 = \lim\limits_{m\to\infty} a_{2m}$ (1pt)

$\{a_n\}$ cannot be convergent

If they indicate that they can apply the right Thm, they may get 2pts.

2. (10 points) Show by definition that $f(x) = |\cos(x)| : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$.

We need to $\forall \xi \in \mathbb{R}$, f is cont. at ξ. 1pt

We WTS: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. 1pt

$|f(x) - f(\xi)| < \varepsilon \quad \forall \ |x - \xi| < \delta$.

Work backwards: $|f(x) - f(\xi)| < \varepsilon$

$\Leftrightarrow \big|\, |\cos(x)| - |\cos(\xi)| \,\big| < \varepsilon$ 2pts

$\Leftarrow |\cos(x) - \cos(\xi)| < \varepsilon$ 1pt

$\Leftrightarrow \left| 2 \sin \frac{x+\xi}{2} \cdot \sin \frac{x-\xi}{2} \right| < \varepsilon$ 1pt

$\Leftarrow 2 \left| \sin \frac{x-\xi}{2} \right| < \varepsilon$ 1pt

$\Leftarrow 2 \left| \frac{x-\xi}{2} \right| < \varepsilon$ 1pt

$\Leftrightarrow |x - \xi| < \varepsilon$

Thus we may take $\delta = \varepsilon$! 2pts

If they didn't use the triangle inequality while everything else are correct, please just take 2pts off & give them all other pts.

3. Consider a function $f(x) = \tan^{-1}(x^2+1) : \mathbb{R} \to \mathbb{R}$.

(a) (4 points) Show that f is continuous on $\mathbb{R}$.

We may write $f(x) = g \circ h(x)$,
where $h(x) = x^2 + 1 : \mathbb{R} \to \mathbb{R}$
and $g(x) = \tan^{-1}(x) : \mathbb{R} \to \mathbb{R}$. (2pts)

Since both h & g are cont. on their domains, f must be cont. on $\mathbb{R}$. (2pts)

Again, if they indicate that they use the right Thm here, give them 2pts.

(b) (6 points) Use the fact from part (1) to compute $\lim_{n\to\infty} \tan^{-1}\left(\frac{1}{n^2}+1\right)$. Justify your steps.

$$\tan^{-1}\left(\frac{1}{n^2}+1\right) = f\left(\frac{1}{n}\right) \quad \text{(2pts)}$$

By cont. of f, we have

$$\lim_{n\to\infty} f\left(\frac{1}{n}\right) = f\left(\lim \frac{1}{n}\right) \quad \text{(2pts)}$$
$$= f(0) \quad \text{(1pt)}$$
$$= \tan^{-1}(1) \quad \text{(1pt)}$$
$$= \frac{\pi}{2}$$

They should get at least 2pts if they somehow show they know which Thm to apply.

4. (10 points) Does $f(x) = \sin\left(\frac{1}{x-1}\right) : (1, \infty) \to \mathbb{R}$ have a right limit at $x = 1$? If yes, compute the limit. If not, explain why it doesn't exist.

$\lim_{x \to 1+} f(x)$ does not exist.

- Set $\frac{1}{x-1} = n\pi \Rightarrow x = \frac{1}{n\pi} + 1$.
Thus by taking $x_n^{(1)} = \frac{1}{n\pi} + 1, n \geq 1$, 2pts
we have $f(x_n^{(1)}) = \sin(n\pi) = 0 \quad \forall n \geq 1$.
$\Rightarrow \lim_{n \to \infty} f(x_n^{(1)}) = 0$ 2pts

- Set $\frac{1}{x-1} = 2n\pi + \frac{\pi}{2} \Rightarrow x = \frac{1}{2n\pi + \frac{\pi}{2}} + 1$.
Thus by taking $x_n^{(2)} = \frac{1}{2n\pi + \frac{\pi}{2}} + 1, n \geq 1$ 2pts
we have $f(x_n^{(2)}) = \sin(2n\pi + \frac{\pi}{2}) = 1 \quad \forall n \geq 1$
$\Rightarrow \lim_{n \to \infty} f(x_n^{(2)}) = 1$ 2pts

Since we have $\lim_{n \to \infty} x_n^{(1)} = \lim_{n \to \infty} x_n^{(2)} = 1$
but $\lim_{n \to \infty} f(x_n^{(1)}) = 0 \neq 1 = \lim_{n \to \infty} f(x_n^{(2)})$,
f cannot have a right limit at $x = 1$. 2pts

If the students somehow indicates that they should use Thm 2.2 here, they should get at least 2pts

5. *In this question, we fix $a > 1$ and you may use the following three facts:*

   1. $a^x : \mathbb{R} \to (0, \infty)$ *is continuous and strictly monotone increasing on* $\mathbb{R}$,
   2. $\log_a(x) : (0, \infty) \to \mathbb{R}$ *is the inverse function of* a^x, *and*
   3. $\log_a(x^y) = y \log_a(x)$ *for all* $x > 0$ *and all* $y \in \mathbb{R}$.

   Consider the following two questions.

   (a) (4 points) Show that $\log_a(x)$ is continuous and strictly monotone increasing on $(0, \infty)$.

We know that inverse function of a strictly monotone increasing function is strictly monotone increasing, $\log_a(x)$ must be strictly monotone increasing. 1pt

For each $x \in (0, \infty)$, we can find a closed bound interval, e.g. $[\frac{1}{n}, n]$ for large $n \in \mathbb{Z}_+$ s.t. $x \in (\frac{1}{n}, n)$. (1pt)

Since $a^x : [\log_a(\frac{1}{n}), \log_a(n)] \to [\frac{1}{n}, n]$ is cont., (1pt)

$\log_a(x)$ is cont. at $x \in (\frac{1}{n}, n)$.

Since x is arbitrary, we get $\log_a(x)$ is cont. on $(0, +\infty)$ (1pt)

They may only get 1pt for showing cont. if they didn't restrict it to a closed, bounded interval.

(b) (6 points) Consider the function $f(x) = x^{\sqrt{2}} : (0, \infty) \to (0, \infty)$. Show that $f(x)$ is continuous and stictly monotone increasing on $(0, \infty)$.
Here you may the conclusion of part (a) as well, even though you are not able to give the full details of part (a).

We have $f(x) = e^{\ln(f(x))}$ since $f(x) > 0$

$= e^{\ln(x^{\sqrt{2}})}$ 2pts

$= e^{\sqrt{2}\ln(x)}$ since $x > 0$

Thus $f(x) = g \circ h(x)$ where 1pt

$h(x) = \sqrt{2}\ln(x)$ is cont. on $(0, +\infty)$ 1pt

and $g(x) = e^x$ is cont. on $\mathbb{R}$. 1pt

Thus as a composition of cont. functions, $f(x)$ is cont. on its domain $(0, +\infty)$. 1pt

They may get 2pts if they indicate they which Thm to use to show cont.

6. (a) (4 points) Consider a subset $S \subset [a, b]$. Suppose S is an infinite set. Is it possible that its limit set S' is empty? Why?

S' cannot be empty.

Because $S \subset [a,b]$ implies S is bounded. (2pts)

S is infinite as well.

Thus by Bolzano-Weierstrass, S has at least one limit point, i.e. $S' \neq \phi$ (2pts)

(b) (6 points) Is there an infinite subset S of $\mathbb{R}$ which has no limit point, i.e. $S' = \varnothing$? If yes, give such an example and show why your S satisfies $S' = \varnothing$. If no, explain your reasoning.

Yes, there are such sets. (1pt)

For instance, we may take $S = \mathbb{Z}$. (1pt)

Indeed $\mathbb{Z}$ is infinite & $\mathbb{Z}' = \phi$.

To show $\mathbb{Z}' = \phi$, we show $\forall \alpha \in \mathbb{R}$, $\alpha \notin \mathbb{Z}'$.

Case I: If $\alpha \in \mathbb{Z}$, then $(\alpha - \frac{1}{2}, \alpha + \frac{1}{2}) \cap (\mathbb{Z} \setminus \{\alpha\}) = \phi$ (2pts)

$\Rightarrow \alpha \notin \mathbb{Z}'$.

Case II: If $\alpha \notin \mathbb{Z}'$, then $\exists\, n \in \mathbb{Z}$ s.t.

$n < \alpha < n+1$. Taking $\varepsilon = \min\{n+1-\alpha, \alpha - n\}$, (2pts)

we have $(\alpha - \varepsilon, \alpha + \varepsilon) \cap \mathbb{Z} = \phi$

$\Rightarrow \alpha \notin \mathbb{Z}'$.

This problem worths 5 extra points. Note this 5 point is not part of your final exam score. It will be added to your standard final letter grade. For this problem, either you have a perfect solution then you will get a 5, or you will get a 0.

Extra credit problem: let $f : I \to \mathbb{R}$ be a continuous function defined on a closed, bounded interval $I = [a, b]$. Assume that $f(a) = f(b)$ and $m < f(a) < M$ where m and M are the minimum and maximum of f on $[a, b]$, respectively. Show that for each $\gamma \in (m, M)$, there are at least two different points $x_1 \neq x_2 \in [a, b]$ such that $f(x_1) = f(x_2) = \gamma$.

We may let $f(\xi_1) = m$ & $f(\xi_2) = M$.

Without loss of generality, we may assume $a < \xi_1 < \xi_2 < b$.

No we take a $\gamma \in (m, M)$.

Case I: If $\gamma = f(a)$, then we have $a \neq b$ s.t. $f(a) = f(b) = \gamma$

Case II: If $\gamma \neq f(a)$, then

(5pts)

- either $\gamma > f(a)$, then $f(\xi_1) < \gamma < f(\xi_2)$ & $f(b) < \gamma < f(\xi_2)$.

  By IVT, there are $x_1 \in (\xi_1, \xi_2)$ & $x_2 \in (\xi_2, b)$ s.t. $f(x_1) = f(x_2) = \gamma$

- or $\gamma < f(a)$, then $f(\xi_1) < \gamma < f(\xi_2)$ & $f(\xi_1) < \gamma < f(a)$

By IVT, there are $x_1 \in (\xi_1, \xi_2)$ & $x_2 \in (a, \xi_1)$ s.t. $f(x_1) = f(x_2) = \gamma$

done.