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Homework 1 solutions

Reminder: If you choose to view these solutions, please only do so whenever you cannot make further progress on solving a problem. Make sure you understand firmly what is being written, and make sure that you are writing your own proof in your own words!

1. Consider the following questions concerning bounded sets.

(1) State the definition of a bounded set $S \subset \mathbb{R}$.

Remark. *See Definition 1.1 in page 10 of Section 1.1 of your professor's lecture notes.*

Definition. A set $S \subset \mathbb{R}$ is said to be *bounded* if there exist $m, M \in \mathbb{R}$ such that $m \leq x \leq M$ for all $x \in S$. □

(2) Show that $S = \{1, \ldots, 10\} \cup (-1, 1)$ is bounded.

Proof. Let $m = -1$ and $M = 10$. Then we have $m \leq x \leq M$ for all $x \in S$. So $S \subset \mathbb{R}$ is bounded. □

(3) Show that the set of natural numbers $\mathbb{N}$ is an unbounded subset of $\mathbb{R}$.

Remark. *See the Claim in page 11 of Section 1.1 of your professor's lecture notes to see the proof showing that $\mathbb{Z}$ is unbounded. The proof to show that $\mathbb{N}$ is unbounded is almost identical to the proof showing that $\mathbb{Z}$ is unbounded.*

Proof. We will use a proof by contradiction. Suppose instead that $\mathbb{N}$ is bounded above by some M. Then we have $n \leq M$ for all $n \in \mathbb{N}$. But we know that, for all $M \in \mathbb{R}$, there exists the least integer greater than or equal to M, which we denote $\lceil M \rceil$. So we have $n \leq M \leq \lceil M \rceil$, which implies that $\lceil M \rceil$ is, in fact, a natural number; that is, $\lceil M \rceil \in \mathbb{N}$. Furthermore, adding a natural number by 1 is again a natural number; in other words, we have $\lceil M \rceil + 1 \in \mathbb{N}$. And we also get

$$\lceil M \rceil + 1 > \lceil M \rceil.$$

At the same time, we assumed earlier that all natural numbers are bounded above by M; that is, we assumed $n \leq M$ for all $n \in \mathbb{N}$. In particular, as we have just showed $\lceil M \rceil + 1 \in \mathbb{N}$, it is bounded above by M; that is, we get

$$\lceil M \rceil + 1 \leq \lceil M \rceil.$$

Therefore, we combine our above two displayed inequalities to conclude

$$\begin{aligned} \lceil M \rceil &< \lceil M \rceil + 1 \\ &\leq \lceil M \rceil, \end{aligned}$$

which is a contradiction. □

(4) Show that the union of two bounded sets is bounded.

Proof. Let $S_1, S_2 \subset \mathbb{R}$ be two bounded sets. Since S_1 is bounded, there exist $m_1, M_1 \in \mathbb{R}$ such that $m_1 \leq x \leq M_1$ for all $x \in S_1$. Since S_2 is bounded, there exist $m_2, M_2 \in \mathbb{R}$ such that $m_2 \leq y \leq M_2$ for all $y \in S_2$. Now, let $z \in S_1 \cup S_2$. Then $z \in S_1$ or $z \in S_2$. Also set $m = \min\{m_1, m_2\}$ and $M = \max\{M_1, M_2\}$.

- Case 1: If $z \in S_1$, then we have $m_1 \leq z \leq M_1$. In fact, we have

$$\begin{aligned} m &= \min\{m_1, m_2\} \\ &\leq m_1 \\ &\leq z \\ &\leq M_1 \\ &\leq \max\{M_1, M_2\} \\ &= M. \end{aligned}$$

- Case 2: If $z \in S_2$, then we have $m_2 \leq z \leq M_2$. In fact, we have

$$\begin{aligned} m &= \min\{m_1, m_2\} \\ &\leq m_2 \\ &\leq z \\ &\leq M_2 \\ &\leq \max\{M_1, M_2\} \\ &= M. \end{aligned}$$

In either case, we have $m \leq z \leq M$ for all $z \in S_1$ or for all $z \in S_2$. In other words, we have $m \leq z \leq M$ for all $z \in S_1 \cup S_2$. Therefore, $S_1 \cup S_2$ is bounded. □

2. The triangle inequality asserts $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

   (1) When does equality hold in triangle inequality?

   Remark. *This required separating this problem into cases, and I proved each one explicitly. However, you do not need to prove every case explicitly! Most of the proofs below involve interchanging the roles of x and y, so they come off as redundant. To shorten your own proof, you can just say "Without loss of generality" for one case, and prove that one case only, provided that the proofs of other cases are similar. Otherwise, if a proof of another case looks substantially different, then it might be worth including that in your own proof.*

   Proof. Equality holds if x, y are both positive, both negative, or at least one of x, y is zero.

   - Case 1: Suppose x, y are both positive. Then we have $|x| = x$ and $|y| = y$. Also, $x + y$ is positive, which means we have $|x + y| = x + y$. Therefore, we have

   $$\begin{aligned}|x+y| &= x+y \\ &= |x| + |y|,\end{aligned}$$

   as desired.
   - Case 2: Suppose x, y are both negative. Then we have $|x| = -x$ and $|y| = -y$. Also, $x + y$ is negative, which means we have $|x + y| = -(x + y)$. Therefore, we have

   $$\begin{aligned}|x+y| &= -(x+y) \\ &= (-x) + (-y) \\ &= |x| + |y|,\end{aligned}$$

   as desired.
   - Case 3: Suppose $x = 0$. Then we also have $|x| = 0$. For all $y \in \mathbb{R}$, we have

   $$\begin{aligned}|x+y| &= |0+y| \\ &= |y| \\ &= 0 + |y| \\ &= |x| + |y|,\end{aligned}$$

   as desired.
   - Case 4: Suppose $y = 0$. Then we also have $|y| = 0$. For all $x \in \mathbb{R}$, we have

   $$\begin{aligned}|x+y| &= |x+0| \\ &= |x| \\ &= |x| + 0 \\ &= |x| + |y|,\end{aligned}$$

   as desired.

   These cases complete the proof. □

   (2) When is the inequality strict in triangle inequality?

   Remark. *Same remark as in part (1). This required separating this problem into cases, and I proved each one explicitly. However, you do not need to prove every case explicitly! Most of the proofs below involve interchanging the roles of x and y, so they come off as redundant. To shorten your own proof, you can just say "Without loss of generality" for one case, and prove that one case only, provided that the proofs of other cases are similar. Otherwise, if a proof of another case looks substantially different, then it might be worth including that in your own proof.*

   Direct proof. Whenever x is positive and y is negative, or x is negative and y is positive. We will prove by cases. (Or assume without loss of generality and prove one case only.)

   - Case 1: Suppose x is positive and y is negative. Then we have $|x| = x$ and $|y| = -y$.
     - Subcase 1: Suppose $x + y \geq 0$. Since y is negative, we have $y < 0$, which is equivalent to $y < -y$. So we have

     $$\begin{aligned}|x+y| &= x+y \\ &< x + (-y) \\ &= |x| + |y|,\end{aligned}$$

     as desired.

- Subcase 2: Suppose $x + y < 0$. Since x is positive, we have $x > 0$, which is equivalent to $x > -x$. So we have

$$\begin{aligned}|x+y| &= x+y \\ &< (-x)+y \\ &= |x|+|y|,\end{aligned}$$

as desired.

- Case 2: Suppose x is negative and y is positive. Then we have $|x| = -x$ and $|y| = y$.
  - Subcase 2: Suppose $x + y \geq 0$. Since x is negative, we have $x < 0$, which is equivalent to $x < -x$. So we have

$$\begin{aligned}|x+y| &= x+y \\ &< (-x)+y \\ &= |x|+|y|,\end{aligned}$$

as desired.

  - Subcase 2: Suppose $x + y < 0$. Since y is positive, we have $y > 0$, which is equivalent to $y > -y$. So we have

$$\begin{aligned}|x+y| &= x+y \\ &< x+(-y) \\ &= |x|+|y|,\end{aligned}$$

as desired.

Both cases complete the proof. □

Proof by contradiction. Whenever x is positive and y is negative, or x is negative and y is positive. We will prove by cases. (Or assume without loss of generality and prove one case only.)

- Case 1: Suppose x is positive and y is negative. Then we have $|x| = x$ and $|y| = -y$. We will prove the strict inequality $|x + y| < |x| + |y|$ by contradiction. Suppose instead we have $|x + y| \geq |x| + |y|$.
  - Subcase 1: Suppose $|x + y| > |x| + |y|$. The triangle inequality states $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$. Therefore, we get

$$\begin{aligned}|x|+|y| &< |x+y| \\ &\leq |x|+|y|,\end{aligned}$$

which is a contradiction.

  - Subcase 2: Suppose $|x + y| = |x| + |y|$.
    * Subsubcase 1: If $x + y \geq 0$, then we have $|x + y| = x + y$. Therefore, we have

$$\begin{aligned}x+(-y) &= |x|+|y| \\ &= |x+y| \\ &= x+y,\end{aligned}$$

from which we can subtract x from both sides to obtain $-y = y$, or equivalently $y = 0$. But we assumed at the beginning of this proof that y is positive. This means we have that y is zero and positive at the same time, which is a contradiction.

    * Subsubcase 2: If $x + y < 0$, then we have $|x + y| = -(x + y)$. Therefore, we have

$$\begin{aligned}x+(-y) &= |x|+|y| \\ &= |x+y| \\ &= -(x+y) \\ &= (-x)+(-y),\end{aligned}$$

from which we can subtract $-y$ from both sides to obtain $x = -x$, or equivalently $x = 0$. But we assumed at the beginning of this proof that x is negative. This means we have that x is zero and negative at the same time, which is a contradiction.

In both Subcase 1 and Subcase 2 above, we obtained a contradiction. So we conclude $|x + y| < |x| + |y|$.

- Case 2: Suppose x is negative and y is positive. Then we have $|x| = -x$ and $|y| = y$. We will prove the strict inequality $|x + y| < |x| + |y|$ by contradiction. Suppose instead we have $|x + y| \geq |x| + |y|$.
  - Subcase 1: Suppose $|x + y| > |x| + |y|$. The triangle inequality states $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$. Therefore, we get

$$\begin{aligned}|x|+|y| &< |x+y| \\ &\leq |x|+|y|,\end{aligned}$$

which is a contradiction.

– Subcase 2: Suppose $|x + y| = |x| + |y|$.
  * Subsubcase 1: If $x + y \geq 0$, then we have $|x + y| = x + y$. Therefore, we have

$$\begin{aligned}(-x) + y &= |x| + |y| \\ &= |x + y| \\ &= x + y,\end{aligned}$$

from which we can subtract y from both sides to obtain $-x = x$, or equivalently $x = 0$. But we assumed at the beginning of this proof that x is negative. This means we have that x is zero and negative at the same time, which is a contradiction.
  * Subsubcase 2: If $x + y < 0$, then we have $|x + y| = -(x + y)$. Therefore, we have

$$\begin{aligned}(-x) + y &= |x| + |y| \\ &= |x + y| \\ &= -(x + y) \\ &= (-x) + (-y),\end{aligned}$$

from which we can subtract $-x$ from both sides to obtain $y = -y$, or equivalently $y = 0$. But we assumed at the beginning of this proof that y is positive. This means we have that y is zero and positive at the same time, which is a contradiction.

In both Subcase 1 and Subcase 2 above, we obtained a contradiction. So we conclude $|x + y| < |x| + |y|$.

These cases complete the proof by contradiction. □

(3) Use the triangle inequality to show that $|x - y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

Proof. The triangle inequality states

$$|x + y| \leq |x| + |y|$$

for all $x, y \in \mathbb{R}$. Using this triangle inequality, we get

$$\begin{aligned}|x - y| &= |x + (-y)| \\ &\leq |x| + |-y| \\ &= |x| + |y|\end{aligned}$$

for all $x, y \in \mathbb{R}$. □

(4) Use the triangle inequality to show that $||x| - |y|| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Remark. *See the Corollary near the top of page 12 in Section 1.1 of your professor's lecture notes to find this proof. The proof below is presented in my own words; it is presented differently from the professor's proof.*

Proof. For all $x, y \in \mathbb{R}$, we have

$$\begin{aligned}|x| - |y| &= |(x - y) + y| - |y| && \text{add and subtract } y \\ &\leq (|x - y| + |y|) - |y| && \text{by part (3) on the first term} \\ &= |x - y|\end{aligned}$$

and

$$\begin{aligned}-(|x| - |y|) &= |y| - |x| \\ &= |(y - x) + x| - |x| && \text{add and subtract } x \\ &\leq (|y - x| + |x|) - |x| && \text{by part (3) on the first term} \\ &= |y - x| \\ &= |x - y|.\end{aligned}$$

These two inequalities imply

$$\begin{aligned}||x| - |y|| &= \pm(|x| - |y|) \\ &\leq |x - y|\end{aligned}$$

for all $x, y \in \mathbb{R}$. □

3. Consider the following questions concerning the definition of a convergent sequence.

(1) State the definition of a convergent sequence.

Remark. *See Definition 1.3 at the top of page 13 in Section 1.2 of your professor's lecture notes.*

Proof. We say that a sequence $\{a_n\}$ *converges* to its limit $L \in \mathbb{R}$ as n goes to ∞ if, for any $\epsilon > 0$, there exists a positive integer n_0 such that $|a_n - L| < \epsilon$ for all $n \geq n_0$. □

(2) Use the definition to show that, if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} |a_n| = |L|$.

Proof. Suppose $\lim_{n\to\infty} a_n = L$. Let $\epsilon > 0$ be given. (This is another way of saying "for any $\epsilon > 0$".) Then there exists a positive integer n_0 such that $|a_n - L| < \epsilon$ for all $n \geq n_0$. Furthermore, by Exercise 2, part (4) (the reverse triangle inequality), we have

$$\begin{aligned} ||a_n| - |L|| &\leq |a_n - L| \\ &< \epsilon. \end{aligned}$$

In other words, for any $\epsilon > 0$, there exists a positive integer n_0 such that $||a_n| - |L|| < \epsilon$ for all $n \geq n_0$. So we conclude $\lim_{n\to\infty} |a_n| = |L|$. □

(3) Is it true that, if $\lim_{n\to\infty} |a_n| = |L|$, then $\lim_{n\to\infty} a_n = L$?

Answer. This is not necessarily true. For our counterexample, consider the sequence $\{a_n\}$ given by

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

for all $n \in Z_+$ and $L = 1$. (Note that we can equivalently write $a_n = (-1)^n$ for all $n \in \mathbb{Z}_+$.) Then we have

$$\begin{aligned} \lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} |(-1)^n| \\ &= \lim_{n\to\infty} 1 \\ &= 1 \\ &= |1| \\ &= |L|. \end{aligned}$$

However, as seen in Example 3 on page 14 of Section 1.2 of the lecture notes, the sequence $\{a_n\}$ is a divergent sequence. Indeed, we can consider two subsequences $\{a_{2k}\}, \{a_{2k-1}\} \subset \{a_n\}$ defined by $a_{2k} = 1$ and $a_{2k-1} = -1$ for all $k \in \mathbb{Z}_+$, which have their respective limits $\lim_{k\to\infty} a_{2k} = 1$ and $\lim_{k\to\infty} a_{2k-1} = -1$. These two limits are different, which implies that the limit of $a_n = (-1)^n$ does not exist. In other words, we cannot write $\lim_{n\to\infty} a_n = L$ for all $L \in \mathbb{R}$. □

4. Mimic the class examples to show, using the definition of a convergent sequence, that the following sequences converge to their limits.

(1) $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Proof. Let $\epsilon > 0$ be given, and choose a positive integer $n_0 > \sqrt{\frac{1}{\epsilon}}$. For all $n \geq n_0$, we have

$$\begin{aligned} \left|\frac{1}{n^2} - 0\right| &= \left|\frac{1}{n^2}\right| \\ &= \frac{1}{n^2} \\ &\leq \frac{1}{n_0^2} \\ &< \frac{1}{\left(\sqrt{\frac{1}{\epsilon}}\right)^2} \\ &= \epsilon. \end{aligned}$$

Therefore, by the definition of a convergent sequence, we conclude $\lim_{n\to\infty} \frac{1}{n^2} = 0$. □

(2) $\lim_{n\to\infty} \frac{1}{2^n} = 0$

Proof. Let $\epsilon > 0$ be given, and choose a positive integer $n_0 > \frac{1}{\epsilon}$. Notice the inequality $2^n \geq n$ for all $n \in \mathbb{Z}_+$. For all $n \geq n_0$, we have

$$\begin{aligned} \left|\frac{1}{2^n} - 0\right| &= \left|\frac{1}{2^n}\right| \\ &= \frac{1}{2^n} \\ &\leq \frac{1}{n} \\ &\leq \frac{1}{n_0} \\ &= \epsilon. \end{aligned}$$

Therefore, by the definition of a convergent sequence, we conclude $\lim_{n\to\infty} \frac{1}{2^n} = 0$. □

(3) $\lim_{n\to\infty} n^2 a^n = 0$ for all $0 < a < 1$

Proof. Since $0 < a < 1$, we can write, for instance,

$$a = \frac{1}{(1+h)^3}$$

for any $h > 0$. Bernoulli's inequality states

$$(1+h)^n \geq 1 + nh$$

for all $n \in \mathbb{Z}_+$ and for all $h > 0$. Consequently, we get

$$\begin{aligned} a^n &= \left(\frac{1}{(1+h)^3}\right)^n \\ &= \frac{1}{((1+h)^n)^3} \\ &\leq \frac{1}{(1+nh)^3} \\ &< \frac{1}{(nh)^3} \\ &= \frac{1}{n^3h^3} \end{aligned}$$

for all $n \in \mathbb{Z}_+$. Now, let $\epsilon > 0$ be given, and for any fixed $h > 0$, choose a positive integer $n_0 > \frac{1}{\epsilon h^3}$. For all $n \geq n_0$, we have

$$\begin{aligned} |n^2a^n - 0| &= |n^2a^n| \\ &= n^2a^n \\ &< n^2\left(\frac{1}{n^3h^3}\right) \\ &= \frac{1}{nh^3} \\ &\leq \frac{1}{n_0h^3} \\ &< \frac{1}{(\frac{1}{\epsilon h^3})h^3} \\ &= \epsilon. \end{aligned}$$

Therefore, by the definition of a convergent sequence, we conclude $\lim_{n\to\infty} n^2a^n = 0$ for all $0 < a < 1$. □