# HOMEWORK ASSIGNMENT TWO

MATH 150A, WINTER 2020

1. Try to mimick the class examples to show convergence of the following sequences by definition.

(1) $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

(2) $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

(3) Let $0 < a < 1$. Show that $\lim_{n\to\infty} n^2 a^n = 0$.

(4) Try this one if you are ambious (**it won't be on the quiz**):
Let $0 < a < 1$ and $k \in \mathbb{Z}_+$. Show that $\lim_{n\to\infty} n^k a^n = 0$.

2. Show that if $\{a_n\}$ is bounded and $\{b_n\}$ converges to 0, then $\lim_{n\to\infty}(a_n b_n) = 0$.

*Remark: This fact may be viewed as a supplementary theorem to Theorem 1.6–the product law. Because here we do not need the convergence of $\{a_n\}$ while we do require that $\lim_{n\to\infty} b_n = 0$.*

3. Consider the following questions.

(1) Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences. Let $\alpha$ and $\beta \in \mathbb{R}$ be two constants. Apply the product and sum laws to show that

$$\lim_{n\to\infty}(\alpha a_n + \beta b_n) = \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n.$$

*Remark: This says that the operation of taking limits is linear. By induction, one can extend this property to: the limit of any linear combination of convergent sequences is the linear combination of the limits.*

(2) Use part (4) of problem 1 and part (1) of porblem 3 (or rather its remark) to show that

$$\lim_{n\to\infty}\big(p(n)a^n\big) = 0,$$

where $p(n) = \sum_{k=0}^{m} a_k n^k$ is a polynomial in $n$ and $0 < a < 1$ is a constant.

*Remark: As we mentioned in class, this fact tells us that exponential decaying ( given by $a^n$) always beats polynomial growth (given by $p(n)$).*

4. Consider the following quesitons:

(1) Find an example where $\{a_n\}$ and $\{b_n\}$ are both divergent, but $\{a_n + b_n\}$ is convergent.

(2) If $\{a_n\}$ is convergent and $\{b_n\}$ is divergent, can $\{a_n + b_n\}$ be convergent? Use the sum law to explain your answer.

*Remark: One can ask similar questions concerning the product of two sequences.*