MATH 150A discussion
University of California, Riverside

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Homework 2 solutions

Reminder: If you choose to view these solutions, please only do so whenever you cannot make further progress on solving a problem. Make sure you understand firmly what is being written, and make sure that you are writing your own proof in your own words!

1. Mimic the class examples to show, using the definition of a convergent sequence, that the following sequences converge to their limits.

(1) $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Proof. Let $\epsilon > 0$ be given, and choose a positive integer $n_0 > \sqrt{\frac{1}{\epsilon}}$. For all $n \geq n_0$, we have

$$\begin{aligned}\left|\frac{1}{n^2} - 0\right| &= \left|\frac{1}{n^2}\right| \\ &= \frac{1}{n^2} \\ &\leq \frac{1}{n_0^2} \\ &< \frac{1}{\left(\sqrt{\frac{1}{\epsilon}}\right)^2} \\ &= \epsilon.\end{aligned}$$

Therefore, by the definition of a convergent sequence, we conclude $\lim_{n\to\infty} \frac{1}{n^2} = 0$. □

(2) $\lim_{n\to\infty} \frac{1}{2^n} = 0$

Proof. Let $\epsilon > 0$ be given, and choose a positive integer $n_0 > \frac{\epsilon}{2}$. One can prove by induction the inequality $2^{n-1} \geq n$ for all $n \in \mathbb{Z}_+$. Note that $2^{n-1} \geq n$ implies $2^n = 2 \cdot 2^{n-1} \geq 2n$. For all $n \geq n_0$, we have

$$\begin{aligned}\left|\frac{1}{2^n} - 0\right| &= \left|\frac{1}{2^n}\right| \\ &= \frac{1}{2^n} \\ &\leq \frac{1}{2n} \\ &\leq \frac{1}{2n_0} \\ &< \epsilon.\end{aligned}$$

Therefore, by the definition of a convergent sequence, we conclude $\lim_{n\to\infty} \frac{1}{2^n} = 0$. □

(3) $\lim_{n\to\infty} n^2 a^n = 0$ for all $0 < a < 1$

Proof. Since $0 < a < 1$, we can write, for instance,

$$a = \frac{1}{(1+h)^3}$$

for any $h > 0$. By the binomial expansion, we have

$$\begin{aligned}(1+h)^n &= \sum_{k=0}^{n} \binom{n}{k} h^k 1^{n-k} \\ &= \binom{n}{0} 1^n h^0 + \binom{n}{1} 1^{n-1} h^1 \\ &\geq 1 + nh\end{aligned}$$

for all $n \in \mathbb{Z}_+$ and for all $h > 0$. Consequently, we get

$$\begin{aligned} a^n &= \left(\frac{1}{(1+h)^3}\right)^n \\ &= \frac{1}{((1+h)^n)^3} \\ &\le \frac{1}{(1+nh)^3} \\ &< \frac{1}{(nh)^3} \\ &= \frac{1}{n^3h^3} \end{aligned}$$

for all $n \in \mathbb{Z}_+$. Now, let $\epsilon > 0$ be given, and for any fixed $h > 0$, choose a positive integer $n_0 > \frac{1}{\epsilon h^3}$. For all $n \ge n_0$, we have

$$\begin{aligned} |n^2a^n - 0| &= |n^2a^n| \\ &= n^2a^n \\ &< n^2\left(\frac{1}{n^3h^3}\right) \\ &= \frac{1}{nh^3} \\ &\le \frac{1}{n_0h^3} \\ &< \frac{1}{(\frac{1}{\epsilon h^3})h^3} \\ &= \epsilon. \end{aligned}$$

Therefore, by the definition of a convergent sequence, we conclude $\lim_{n\to\infty} n^2a^n = 0$ for all $0 < a < 1$. □

(4) Let $0 < a < 1$ and $k \in \mathbb{Z}_+$. Show that $\lim_{n\to\infty} n^k a^n = 0$.

Direct proof. Since $0 < a < 1$, we can write, for instance,

$$a = \frac{1}{(1+h)^{k+1}}$$

for any $h > 0$ and for all positive integers k. The binomial expansion implies

$$\begin{aligned} (1+h)^n &= \sum_{k=0}^{n} \binom{n}{k} h^k 1^{n-k} \\ &\ge \binom{n}{0} h^0 1^{n-0} + \binom{n}{1} h^1 1^{n-1} \\ &= 1 + nh \end{aligned}$$

for all $n \in \mathbb{Z}_+$ and for all $h > 0$. Consequently, we get

$$\begin{aligned} a^n &= \left(\frac{1}{(1+h)^{k+1}}\right)^n \\ &= \frac{1}{((1+h)^n)^{k+1}} \\ &\le \frac{1}{(1+nh)^{k+1}} \\ &< \frac{1}{(nh)^{k+1}} \\ &= \frac{1}{n^{k+1}h^{k+1}} \end{aligned}$$

for all $n \in \mathbb{Z}_+$. Now, let $\epsilon > 0$ be given, and for any fixed $h > 0$, choose a positive integer $n_0 > \frac{1}{\epsilon h^{k+1}}$. For all $n \ge n_0$, we have

$$\begin{aligned}
|n^k a^n - 0| &= |n^k a^n| \\
&= n^k a^n \\
&< n^k \left(\frac{1}{n^{k+1} h^{k+1}} \right) \\
&= \frac{1}{n h^{k+1}} \\
&\leq \frac{1}{n_0 h^{k+1}} \\
&< \frac{1}{(\frac{1}{\epsilon h^{k+1}}) h^{k+1}} \\
&= \epsilon.
\end{aligned}$$

Therefore, by the definition of a convergent sequence, we conclude $\lim\limits_{n\to\infty} n^k a^n = 0$ for all $0 < a < 1$ and for all positive integers k. □

Proof by induction. To prove the base case, we would need to show that the statement for $n = 1$—that is, $\lim\limits_{n\to\infty} na^n = 0$ for all $0 < a < 1$—holds true. But you can just follow the argument of part (3), though you need to make the necessary adjustments. (I am leaving that part up to you to do it yourself.) So now it remains to complete the induction step. Assume that the statement holds true for $n = k$—that is, assume $\lim\limits_{k\to\infty} n^k a^n = 0$—for any chosen integer $k \geq 2$. We will prove that the statement holds true for $n = k + 1$. By applying the Quotient Law (Theorem 1.8 of the professor's lecture notes), we observe that, for all $0 < a < 1$,

$$\begin{aligned}
\lim_{n\to\infty} n^{k+1} a^n &= \lim_{n\to\infty} \frac{n^{k+1}}{a^{-n}} \\
&= \frac{\lim\limits_{n\to\infty} n^{k+1}}{\lim\limits_{n\to\infty} a^{-n}} \\
&= \frac{\infty}{\infty}
\end{aligned}$$

is an indeterminate form. So we can apply l'Hôpital's rule to obtain

$$\begin{aligned}
\lim_{n\to\infty} n^{k+1} a^n &= \lim_{n\to\infty} \frac{n^{k+1}}{a^{-n}} \\
&= \lim_{n\to\infty} \frac{\frac{d}{dn} n^{k+1}}{\frac{d}{dn} a^{-n}} \\
&= \lim_{n\to\infty} \frac{(k+1)n^k}{-a^{-n} \ln a} \\
&= -\frac{k+1}{\ln a} \lim_{n\to\infty} \frac{n^k}{a^{-n}} \\
&= -\frac{k+1}{\ln a} \lim_{n\to\infty} n^k a^n \\
&= -\frac{k+1}{\ln a} \cdot 0 \\
&= 0,
\end{aligned}$$

which means the statement holds true for $n = k + 1$. This completes the proof by induction. □

2. Show that if $\{a_n\}_{n=1}^{\infty}$ is bounded and $\{b_n\}_{n=1}^{\infty}$ converges to 0, then $\lim\limits_{n\to\infty} (a_n b_n) = 0$.

Remark (from the professor)**.** *This fact may be viewed as a supplementary theorem to Theorem 1.6 (the product law). Because here we do not need the convergence of* $\{a_n\}_{n=1}^{\infty}$ *while we do require that* $\lim\limits_{n\to\infty} b_n = 0$.

Proof. Since we assume that $\{a_n\}_{n=1}^{\infty}$ is a sequence of bounded terms a_n, there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{Z}_+$. Now, let $\epsilon > 0$ be given. Since we assume that $\lim\limits_{n\to\infty} b_n = 0$, there exists $n_0 \in \mathbb{Z}_+$ such that $|b_n - 0| < \frac{\epsilon}{M}$ for all $n \geq n_0$.

So we have

$$\begin{aligned}
|a_n b_n - 0| &= |a_n b_n| \\
&= |a_n||b_n| \\
&= |a_n||b_n - 0| \\
&\le M|b_n - 0| \\
&< M\left(\frac{\epsilon}{M}\right) \\
&= \epsilon.
\end{aligned}$$

Therefore, we conclude $\lim_{n\to\infty}(a_n b_n) = 0$. □

3. Consider the following questions.

   (1) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two convergent sequences. Let $\alpha, \beta \in \mathbb{R}$ be two constants. Apply the product and sum laws to show that
   $$\lim_{n\to\infty}(\alpha a_n + \beta b_n) = \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n.$$

   Remark (from the professor)**.** *This says that the operation of taking limits is linear. By induction, one can extend this property to: the limit of any linear combination of convergent sequences is the linear combination of the limits.*

   Proof. By the Corollary of the Product Law, if $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence and $c \in \mathbb{R}$ is a constant, then $\{ca_n\}_{n=1}^{\infty}$ is also a convergent sequence that satisfies
   $$\lim_{n\to\infty}(ca_n) = c \lim_{n\to\infty} a_n.$$
   In particular, since $\alpha, \beta \in \mathbb{R}$ are two constants, $\{\alpha a_n\}_{n=1}^{\infty}, \{\beta b_n\}_{n=1}^{\infty}$ are two convergent sequences that satisfy
   $$\begin{aligned}
   \lim_{n\to\infty}(\alpha a_n) &= \alpha \lim_{n\to\infty} a_n, \\
   \lim_{n\to\infty}(\beta b_n) &= \beta \lim_{n\to\infty} b_n.
   \end{aligned}$$
   Then the Sum Law asserts that $\{\alpha a_n + \beta b_n\}_{n=1}^{\infty}$ is a convergent sequence that satisfies
   $$\lim_{n\to\infty}(\alpha a_n + \beta b_n) = \lim_{n\to\infty}(\alpha a_n) + \lim_{n\to\infty}(\beta b_n).$$
   Therefore, we have
   $$\begin{aligned}
   \lim_{n\to\infty}(\alpha a_n + \beta b_n) &= \lim_{n\to\infty}(\alpha a_n) + \lim_{n\to\infty}(\beta b_n) \\
   &= \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n,
   \end{aligned}$$
   as desired. □

   (2) Use part (4) of Exercise 1 and part (1) of Exercise 3 (or rather its remark) to show that
   $$\lim_{n\to\infty}(p(n)a^n) = 0,$$
   where $p(n) = \sum_{k=0}^{m} a_k n^k$ is a polynomial in n and $0 < a < 1$ is a constant.

   Remark (from the professor)**.** *As we mentioned in class, this fact tells us that exponential decaying (given by a^n) always beats polynomial growth (given by $p(n)$).*

   Proof. By applying part (1) of Exercise 3 repeatedly (or prove formally by induction), we obtain
   $$\begin{aligned}
   \left(\sum_{k=0}^{m} a_k n^k\right) a^n &= (a_0 + a_1 n + a_2 n^2 + \cdots + a_m n^m)a^n \\
   &= a_0 a^n + a_1 n a^n + a_2 n^2 a^n + \cdots + a_m n^m a^n \\
   &= \sum_{k=0}^{m} a_k n^k a^n
   \end{aligned}$$
   Part (4) of Exercise 1 states
   $$\lim_{n\to\infty}(n^k a^n) = 0$$

for all $k \in \mathbb{Z}_+$. Therefore, we have

$$\begin{aligned}
\lim_{n\to\infty} (p(n)a^n) &= \lim_{n\to\infty} \left(\sum_{k=0}^{m} a_k n^k\right) a^n \\
&= \lim_{n\to\infty} \sum_{k=0}^{m} a_k n^k a^n \\
&= \sum_{k=0}^{m} a_k \lim_{n\to\infty} (n^k a^n) \\
&= \sum_{k=0}^{m} a_k \cdot 0 \\
&= 0,
\end{aligned}$$

as desired. □

4. Consider the following questions:

   (1) Find an example where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are both divergent, but $\{a_n + b_n\}_{n=1}^{\infty}$ is convergent.

   Remark. *What I showed below is just one example that works. There are, of course, infinitely many examples that also answer this question.*

   Proof. Define the two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ by

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

   and

$$b_n = \begin{cases} -1 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

   Then $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are both divergent sequences. However, $\{a_n + b_n\}_{n=1}^{\infty}$ is then given by

$$\begin{aligned}
a_n + b_n &= \begin{cases} (-1) + 1 & \text{if } n \text{ is even,} \\ 1 + (-1) & \text{if } n \text{ is odd.} \end{cases} \\
&= \begin{cases} 0 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \\
&= 0
\end{aligned}$$

   for all $n \in \mathbb{Z}_+$. Furthermore, we have

$$\begin{aligned}
\lim_{n\to\infty} (a_n + b_n) &= \lim_{n\to\infty} 0 \\
&= 0,
\end{aligned}$$

   meaning that the limit of $a_n + b_n$ exists, and so $\{a_n + b_n\}_{n=1}^{\infty}$ is a convergent sequence. □

   (2) If $\{a_n\}_{n=1}^{\infty}$ is convergent and $\{b_n\}_{n=1}^{\infty}$ is divergent, can $\{a_n + b_n\}_{n=1}^{\infty}$ be convergent? Use the sum law to explain your answer.

   Proof. No, if $\{a_n\}_{n=1}^{\infty}$ is convergent and $\{b_n\}_{n=1}^{\infty}$ is divergent, then $\{a_n + b_n\}_{n=1}^{\infty}$ cannot be convergent. To prove this, we will argue by contradiction. Suppose instead that $\{a_n + b_n\}_{n=1}^{\infty}$ is convergent. Since $\{a_n\}_{n=1}^{\infty}$ is convergent, the Corollary of the Product Law (with $c = -1$) asserts that $\{-a_n\}_{n=1}^{\infty}$ is also convergent. Now, we can write

$$b_n = (a_n + b_n) + (-a_n)$$

   for all $n \in \mathbb{Z}_+$; in other words, we can write

$$\{b_n\}_{n=1}^{\infty} = \{(a_n + b_n) + (-a_n)\}_{n=1}^{\infty},$$

   which is the addition of the two convergent sequences $\{a_n + b_n\}_{n=1}^{\infty}$ and $\{-a_n\}_{n=1}^{\infty}$. By the Sum Law, we conclude that $\{b_n\}_{n=1}^{\infty}$ is a convergent sequence. But this contradicts our assumption that $\{b_n\}_{n=1}^{\infty}$ is divergent. □

Remark (from the professor)**.** *One can ask similar questions concerning the product of two sequences.*