MATH 150A discussion
University of California, Riverside

Ryan Ta
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Homework 3 solutions

Reminder: If you choose to view these solutions, please only do so whenever you cannot make further progress on solving a problem. Make sure you understand firmly what is being written, and make sure that you are writing your own proof in your own words!

1. Try to write down the proof of the sum and product limit laws without consulting any resources.

Remark. *The Sum Law and Product Law are Theorem 1.4 and Theorem 1.6 of the professor's lecture notes, respectively.*

Theorem (Sum Law)**.** *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two convergent sequences. Then* $\{a_n + b_n\}_{n=1}^{\infty}$ *is convergent and satisfies*

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Proof. Let $\epsilon > 0$ be given, and let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$ be the respective limits of the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$. Since we assume $a = \lim_{n\to\infty} a_n$, there exists $n_1 > 0$ such that $|a_n - a| < \frac{\epsilon}{2}$ for all $n \geq n_1$. Since we assume $b = \lim_{n\to\infty} b_n$, there exists $n_2 > 0$ such that $|b_n - b| < \frac{\epsilon}{2}$ for all $n \geq n_2$. Now let $n_0 = \max\{n_1, n_2\}$. Then we have

$$\begin{aligned}
|(a_n + b_n) - (a + b)| &= |a_n + b_n - a - b| \\
&= |a_n - a + b_n - b| \\
&= |(a_n - a) + (b_n - b)| \\
&\leq |a_n - a| + |b_n - b| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

So we have proved $\lim_{n\to\infty}(a_n + b_n) = a + b$. In other words, we have

$$\begin{aligned}
\lim_{n\to\infty}(a_n + b_n) &= a + b \\
&= \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n,
\end{aligned}$$

as desired. □

Theorem (Product Law)**.** *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two convergent sequences. Then* $\{a_n b_n\}_{n=1}^{\infty}$ *is convergent and satisfies*

$$\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$

Proof. Let $\epsilon > 0$ be given, and let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$ be the respective limits of the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$. Since we assume $a = \lim_{n\to\infty} a_n$, there exists $n_1 > 0$ such that $|a_n - a| < \frac{\epsilon}{2|b|+1}$ for all $n \geq n_1$. Since we assume $b = \lim_{n\to\infty} b_n$, there exists $n_2 > 0$ such that $|b_n - b| < \frac{\epsilon}{2M}$ for all $n \geq n_2$. Finally, according to Theorem 1.5 of the professor's lecture notes, since $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence, it is also bounded; there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all positive integers n. Now let $n_0 = \max\{n_1, n_2\}$. Then we have

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\
&= |a_n(b_n - b) + (a_n - a)b| \\
&\leq |a_n(b_n - b)| + |(a_n - a)b| \\
&\leq |a_n||b_n - b| + |a_n - a||b| \\
&\leq M|b_n - b| + |a_n - a||b| \\
&< M\frac{\epsilon}{2M} + \frac{\epsilon}{2|b| + 1}|b| \\
&= \epsilon.
\end{aligned}$$

So we have proved $\lim_{n\to\infty}(a_n b_n) = ab$. In other words, we have

$$\begin{aligned}
\lim_{n\to\infty}(a_n b_n) &= ab \\
&= (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n),
\end{aligned}$$

as desired. □

2. Determine whether the following sequences are convergent. If a sequence is convergent, compute its limit and justify your steps. If the sequence is divergent, explain your reasoning. You may directly use the class examples or the examples from previous homework assignments.

(a) $\left\{\dfrac{n3^n + n^2 2^n + 5n}{5^n + n^3}\right\}_{n=1}^{\infty}$

Proof. This sequence is convergent, and we will find its limit. By multiplying and dividing by $(\frac{1}{5})^n$, we obtain

$$\begin{aligned}\frac{n3^n + n^2 2^n + 5n}{5^n + n^3} &= \frac{n3^n + n^2 2^n + 5n}{5^n + n^3} \frac{(\frac{1}{5})^n}{(\frac{1}{5})^n} \\ &= \frac{n(\frac{3}{5})^n + n^2(\frac{2}{5})^n + 5n(\frac{1}{5})^n}{1 + n^3(\frac{1}{5})^n}.\end{aligned}$$

Now, we recall Exercise 1, part (4), of Homework 2, which states that, if $0 < a < 1$ and $k \in \mathbb{Z}_+$, then $\lim\limits_{n\to\infty} n^k a^n = 0$. Invoking that exercise, we have in particular the following:

$$\begin{aligned}\lim_{n\to\infty} n\left(\frac{3}{5}\right)^n &= 0, \\ \lim_{n\to\infty} n^2\left(\frac{2}{5}\right)^n &= 0, \\ \lim_{n\to\infty} n\left(\frac{1}{5}\right)^n &= 0, \\ \lim_{n\to\infty} n^3\left(\frac{1}{5}\right)^n &= 0.\end{aligned}$$

Therefore, using the Sum, Product, and Quotient Laws, we see that the limit is

$$\begin{aligned}\lim_{n\to\infty} \frac{n3^n + n^2 2^n + 5n}{5^n + n^3} &= \lim_{n\to\infty} \frac{n(\frac{3}{5})^n + n^2(\frac{2}{5})^n + 5n(\frac{1}{5})^n}{1 + n^3(\frac{1}{5})^n} \\ &= \frac{\lim\limits_{n\to\infty} (n(\frac{3}{5})^n + n^2(\frac{2}{5})^n + 5n(\frac{1}{5})^n)}{\lim\limits_{n\to\infty} (1 + n^3(\frac{1}{5})^n)} \\ &= \frac{\lim\limits_{n\to\infty} n(\frac{3}{5})^n + \lim\limits_{n\to\infty} n^2(\frac{2}{5})^n + 5 \lim\limits_{n\to\infty} n(\frac{1}{5})^n}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} n^3(\frac{1}{5})^n} \\ &= \frac{0^+0^+5(0)}{1+0} \\ &= 0,\end{aligned}$$

as desired. □

(b) $\{\sqrt{n+2} - \sqrt{n}\}_{n=1}^{\infty}$

Proof. This sequence is convergent, and we will find its limit. By multiplying and dividing by $\sqrt{n+2} + \sqrt{n}$, which is the conjugate of $\sqrt{n+2} - \sqrt{n}$, we obtain

$$\begin{aligned}\sqrt{n+2} - \sqrt{n} &= (\sqrt{n+2} - \sqrt{n}) \frac{\sqrt{n+2} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n}} \\ &= \frac{(n+2) - n}{\sqrt{n+2} + \sqrt{n}} \\ &= \frac{2}{\sqrt{n+2} + \sqrt{n}} \\ &= \frac{2}{\sqrt{n+2} + \sqrt{n}} \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \\ &= \frac{\frac{2}{\sqrt{n}}}{\sqrt{1 + \frac{2}{n}} + 1}.\end{aligned}$$

Next, we observe

$$1 \leq \sqrt{1 + \frac{2}{n}} \leq 1 + \frac{2}{n}$$

for all $n \in \mathbb{Z}_+$. Since $\lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty}(1 + \frac{2}{n}) = 1$, we can use Theorem 1.9 of the professor's lecture notes (Squeeze Theorem) to conclude $\lim_{n\to\infty} \sqrt{1 + \frac{2}{n}} = 1$. Therefore, using the Quotient and Sum Laws, we see that the limit is

$$\begin{aligned}
\lim_{n\to\infty}(\sqrt{n+2} - \sqrt{n}) &= \lim_{n\to\infty} \frac{\frac{2}{\sqrt{n}}}{\sqrt{1+\frac{2}{n}} + 1} \\
&= \frac{\lim_{n\to\infty} \frac{2}{\sqrt{n}}}{\lim_{n\to\infty}(\sqrt{1+\frac{2}{n}} + 1)} \\
&= \frac{\lim_{n\to\infty} \frac{2}{\sqrt{n}}}{\lim_{n\to\infty}\sqrt{1+\frac{2}{n}} + \lim_{n\to\infty} 1} \\
&= \frac{0}{1+1} \\
&= 0,
\end{aligned}$$

as desired. □

(c) $\left\{\dfrac{n^2+1}{n}\right\}_{n=1}^{\infty}$

Proof. For all $n \in \mathbb{Z}_+$, we can rewrite

$$\begin{aligned}
\frac{n^2+1}{n} &= \frac{n^2}{n} + \frac{1}{n} \\
&= n + \frac{1}{n} \\
&\geq n.
\end{aligned}$$

This signifies that the sequence $\left\{\frac{n^2+1}{n}\right\}_{n=1}^{\infty}$ is unbounded, and so it is divergent. □

3. Use the class example $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$ and the formula

$$1 + \frac{3}{n} = \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n+1}\right)\left(1 + \frac{1}{n+2}\right)$$

to show

$$\lim_{n\to\infty}\left(1 + \frac{3}{n}\right)^n = e^3.$$

Proof. First, we will work with the factor $1 + \frac{1}{n+1}$. Let $\ell = n + 1$. Then $n \to \infty$ implies $\ell \to \infty$, and by the Quotient Law and the class example $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\left(1 + \frac{1}{n+1}\right)^n &= \lim_{n\to\infty} \frac{(1 + \frac{1}{n+1})^{n+1}}{1 + \frac{1}{n+1}} \\
&= \frac{\lim_{n\to\infty}(1 + \frac{1}{n+1})^{n+1}}{\lim_{n\to\infty}(1 + \frac{1}{n+1})} \\
&= \frac{\lim_{\ell\to\infty}(1 + \frac{1}{\ell})^{\ell}}{\lim_{\ell\to\infty}(1 + \frac{1}{\ell})} \\
&= \frac{e}{1} \\
&= e.
\end{aligned}$$

Next, we will work with the factor $1 + \frac{1}{n+2}$. Let $m = n + 2$. Then $n \to \infty$ implies $m \to \infty$, and by the Quotient Law and the class example $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^n &= \lim_{n\to\infty}\frac{(1+\frac{1}{n+2})^{n+2}}{(1+\frac{1}{n+1})^2} \\
&= \frac{\lim_{n\to\infty}(1+\frac{1}{n+2})^{n+2}}{\lim_{n\to\infty}(1+\frac{1}{n+2})^2} \\
&= \frac{\lim_{m\to\infty}(1+\frac{1}{m})^{m}}{\lim_{m\to\infty}(1+\frac{1}{m})^2} \\
&= \frac{e}{1} \\
&= e.
\end{aligned}$$

Therefore, by the Product Law, we conclude

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n &= \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)\right)^n \\
&= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n+1}\right)^n\left(1+\frac{1}{n+2}\right)^n \\
&= \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^n\right) \\
&= (e)(e)(e) \\
&= e^3,
\end{aligned}$$

as desired. □

4. Define a sequence by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$ for all integers $n \geq 1$. Prove by induction that $\{a_n\}_{n=1}^{\infty}$ is monotone increasing and bounded above by 2 so that it's convergent. Compute its limit.

Proof. First, we will prove by induction showing that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2; that is, $a_n < 2$ for all positive integers n. The base case is straightforward: for $n = 1$, we have $a_1 = \sqrt{2} < 2$. For the induction step, we assume that the statement for $n = k$ is true: $a_k < 2$, and we will prove that the statement for $n = k+1$ is true. Indeed, the square root function is an increasing function (that is, if $x, y \in \mathbb{R}$ satisfies $x < y$, then $\sqrt{x} < \sqrt{y}$), which allows us to say

$$\begin{aligned}
a_{k+1} &= \sqrt{2+a_k} \\
&< \sqrt{2+2} \\
&= \sqrt{4} \\
&= 2.
\end{aligned}$$

This completes the proof by induction showing that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2. Next, we will prove by induction that $\{a_n\}_{n=1}^{\infty}$ is monotone increasing. For the base case, we have

$$\begin{aligned}
a_1 &= \sqrt{2} \\
&\leq \sqrt{2+\sqrt{2}} \\
&= \sqrt{2+a_1} \\
&= a_2.
\end{aligned}$$

For the induction step, we assume that the statement for $n = k$ is true: $a_k \leq a_{k+1}$, and we will prove that the statement for $n = k+1$ is true. Indeed, because the square root function is an increasing function, inequality is preserved (the inequality sign does not flip), and so we have

$$\begin{aligned}
a_{k+1} &= \sqrt{2+a_k} \\
&\leq \sqrt{2+a_{k+1}} \\
&= a_{k+2},
\end{aligned}$$

as desired. This completes the proof by induction that $\{a_n\}_{n=1}^{\infty}$ is monotone increasing. So we have established that $\{a_n\}_{n=1}^{\infty}$ is both monotone increasing and bounded above by 2. By Theorem 1.11 of the professor's lecture notes (Monotone Convergence

Theorem), we conclude that $\{a_n\}_{n=1}^{\infty}$ is convergent. Finally, we will compute the limit of $\{a_n\}_{n=1}^{\infty}$. Let $L = \lim_{n\to\infty} a_n$ be the limit of $\{a_n\}_{n=1}^{\infty}$. Then the recursive relation $a_{n+1} = \sqrt{2 + a_n}$ and the Sum and Product Laws imply

$$\begin{aligned}
L^2 &= (\lim_{n\to\infty} a_{n+1})^2 \\
&= \lim_{n\to\infty} a_{n+1}^2 \\
&= \lim_{n\to\infty} (\sqrt{2 + a_n})^2 \\
&= \lim_{n\to\infty} (2 + a_n) \\
&= \lim_{n\to\infty} 2 + \lim_{n\to\infty} a_n \\
&= 2 + L,
\end{aligned}$$

from which we obtain the quadratic equation $L^2 - L - 2 = 0$. Solving this quadratic equation, we obtain the solutions $L = 2$ and $L = -1$. However, $L = -1$ is not the limit of $\{a_n\}_{n=1}^{\infty}$ because the sequence is increasing and the first term a_1 is positive, implying that every term a_n is positive, and so $\lim_{n\to\infty} a_n$ is positive, assuming the limit exists. So we conclude that $L = 2$ is the limit of $\{a_n\}_{n=1}^{\infty}$; in other words, we conclude $\lim_{n\to\infty} a_n = 2$. □